A_short_study_of_Renyi_entropy

November 22, 2018

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2 A short study of Rényi entropy

I want to study here the Rényi entropy, using Python. I will define a function implementing $H_\alpha(X)$, from the given formula, for discrete random variables, and check the influence of the parameter α,

$$H_\alpha(X) := \frac{1}{1-\alpha} \log_2(\sum_i^n p_i^\alpha),$$

where X has n possible values, and the i-th outcome has probability $p_i \in [0, 1]$.

- *Reference*: this blog post by John D. Cook, this Wikipédia page and this page on MathWorld,
- *Author*: Lilian Besson
- *License*: MIT License
- *Date*: 22th of November, 2018

2.1 Requirements

```
In [4]: !pip install watermark matplotlib numpy
```

```
Requirement already satisfied: watermark in /usr/local/lib/python3.6/dist-packages (1.5.0)
Requirement already satisfied: matplotlib in /usr/local/lib/python3.6/dist-packages (3.0.2)
Requirement already satisfied: numpy in /usr/local/lib/python3.6/dist-packages (1.14.5)
Requirement already satisfied: ipython in /usr/local/lib/python3.6/dist-packages (from waterma
Requirement already satisfied: cycler>=0.10 in /usr/local/lib/python3.6/dist-packages (from ma
Requirement already satisfied: pyparsing!=2.0.4,!=2.1.2,!=2.1.6,>=2.0.1 in /usr/local/lib/pyth
Requirement already satisfied: python-dateutil>=2.1 in /usr/local/lib/python3.6/dist-packages
Requirement already satisfied: kiwisolver>=1.0.1 in /usr/local/lib/python3.6/dist-packages (fr
Requirement already satisfied: setuptools>=18.5 in /usr/local/lib/python3.6/dist-packages (fro
```

```
Requirement already satisfied: prompt-toolkit<2.1.0,>=2.0.0 in /usr/local/lib/python3.6/dist-pa
Requirement already satisfied: traitlets>=4.2 in /usr/local/lib/python3.6/dist-packages (from i
Requirement already satisfied: pygments in /usr/local/lib/python3.6/dist-packages (from ipythor
Requirement already satisfied: pickleshare in /usr/local/lib/python3.6/dist-packages (from ipyt
Requirement already satisfied: decorator in /usr/local/lib/python3.6/dist-packages (from ipytho
Requirement already satisfied: jedi>=0.10 in /usr/local/lib/python3.6/dist-packages (from ipyth
Requirement already satisfied: pexpect; sys_platform != "win32" in /usr/local/lib/python3.6/dis
Requirement already satisfied: simplegeneric>0.8 in /usr/local/lib/python3.6/dist-packages (fro
Requirement already satisfied: backcall in /usr/local/lib/python3.6/dist-packages (from ipythor
Requirement already satisfied: six in /home/lilian/.local/lib/python3.6/site-packages (from cyc
Requirement already satisfied: wcwidth in /usr/local/lib/python3.6/dist-packages (from prompt-t
Requirement already satisfied: ipython-genutils in /usr/local/lib/python3.6/dist-packages (fror
Requirement already satisfied: parso>=0.3.0 in /usr/local/lib/python3.6/dist-packages (from jec
Requirement already satisfied: ptyprocess>=0.5 in /usr/local/lib/python3.6/dist-packages (from
```

```
In [5]: %load_ext watermark
        %watermark -v -m -a "Lilian Besson" -g -p matplotlib,numpy

The watermark extension is already loaded. To reload it, use:
  %reload_ext watermark
Lilian Besson

CPython 3.6.6
IPython 7.0.1

matplotlib 3.0.2
numpy 1.14.5

compiler   : GCC 8.0.1 20180414 (experimental) [trunk revision 259383
system     : Linux
release    : 4.15.0-38-generic
machine    : x86_64
processor  : x86_64
CPU cores  : 4
interpreter: 64bit
Git hash   : a119f96f2de5b449131a73b6c9861f26b2c0d3f8
```

```
In [7]: import numpy as np
        import matplotlib.pyplot as plt
```

2.2 Utility functions

We start by giving three examples of such vectors $X = (p_i)_{1 \leq i \leq n}$, a discrete probability distributions on n values.

```
In [49]: X1 = [0.25, 0.5, 0.25]
         X2 = [0.1, 0.25, 0.3, 0.45]
         X3 = [0, 0.5, 0.5]

         X4 = np.full(100, 1/100)
         X5 = np.full(1000, 1/1000)

         X6 = np.arange(100, dtype=float)
         X6 /= np.sum(X6)
```

We need a function to safely compute $x \mapsto x \log_2(x)$, with special care in case $x = 0$. This one will accept a numpy array or a single value as argument:

```
In [50]: np.seterr(all="ignore")
```

```
Out[50]: {'divide': 'ignore', 'over': 'ignore', 'under': 'ignore', 'invalid': 'ignore'}
```

```
In [51]: def x_log2_x(x):
             """ Return x * log2(x) and 0 if x is 0."""
             results = x * np.log2(x)
             if np.size(x) == 1:
                 if np.isclose(x, 0.0):
                     results = 0.0
             else:
                 results[np.isclose(x, 0.0)] = 0.0
             return results
```

For examples:

```
In [52]: x_log2_x(0)
         x_log2_x(0.5)
         x_log2_x(1)
         x_log2_x(2)
         x_log2_x(10)
```

```
Out[52]: 0.0
```

```
Out[52]: -0.5
```

```
Out[52]: 0.0
```

```
Out[52]: 2.0
```

```
Out[52]: 33.219280948873624
```

and with vectors, slots with $p_i = 0$ are handled without error:

```
In [54]: x_log2_x(X1)
         x_log2_x(X2)
         x_log2_x(X3)

         x_log2_x(X4)[:10]
         x_log2_x(X5)[:10]
         x_log2_x(X6)[:10]
```

```
Out[54]: array([-0.5, -0.5, -0.5])

Out[54]: array([-0.33219281, -0.5       , -0.52108968, -0.51840139])

Out[54]: array([ 0. , -0.5, -0.5])

Out[54]: array([-0.06643856, -0.06643856, -0.06643856, -0.06643856, -0.06643856,
                -0.06643856, -0.06643856, -0.06643856, -0.06643856, -0.06643856])

Out[54]: array([-0.00996578, -0.00996578, -0.00996578, -0.00996578, -0.00996578,
                -0.00996578, -0.00996578, -0.00996578, -0.00996578, -0.00996578])

Out[54]: array([ 0.        , -0.00247944, -0.00455483, -0.00647773, -0.00830159,
                -0.0100518 , -0.01174333, -0.01338606, -0.01498701, -0.01655143])
```

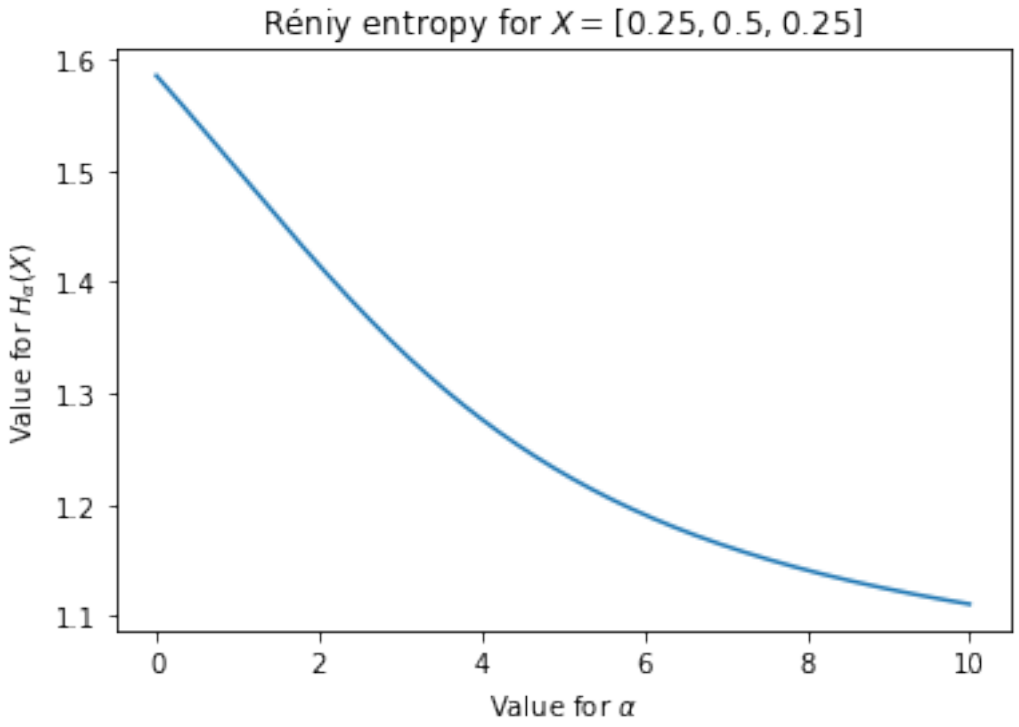
2.3 Definition, common and special cases

From the mathematical definition, an issue will happen if $\alpha = 1$ or $\alpha = \text{inf}$, so we deal with the special cases manually. X is here given as the vector of $(p_i)_{1 \leq i \leq n}$.

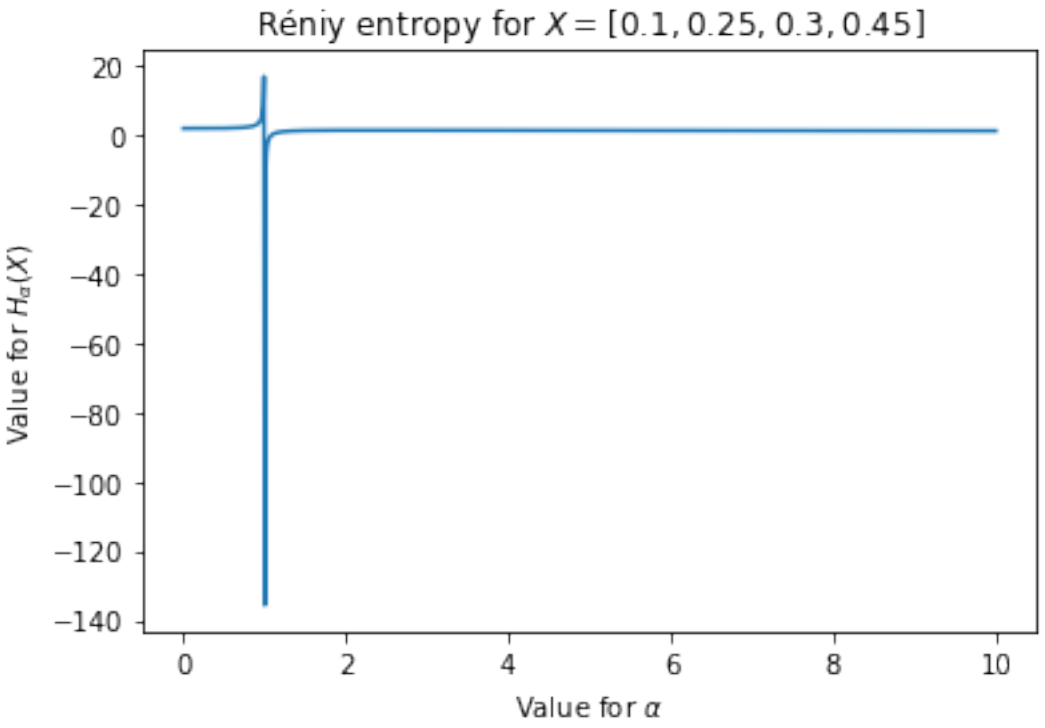
```
In [39]: def renyi_entropy(alpha, X):
             assert alpha >= 0, "Error: renyi_entropy only accepts values of alpha >= 0, but al
             if np.isinf(alpha):
                 # XXX Min entropy!
                 return - np.log2(np.max(X))
             elif np.isclose(alpha, 0):
                 # XXX Max entropy!
                 return np.log2(len(X))
             elif np.isclose(alpha, 1):
                 # XXX Shannon entropy!
                 return - np.sum(x_log2_x(X))
             else:
                 return (1.0 / (1.0 - alpha)) * np.log2(np.sum(X ** alpha))

In [40]: # Curryfied version
         def renyi_entropy_2(alpha):
             def re(X):
                 return renyi_entropy(alpha, X)
             return re

In [42]: # Curryfied version
         def renyi_entropy_3(alphas, X):
             res = np.zeros_like(alphas)
             for i, alpha in enumerate(alphas):
                 res[i] = renyi_entropy(alpha, X)
             return res
```

2.4 Plotting some values

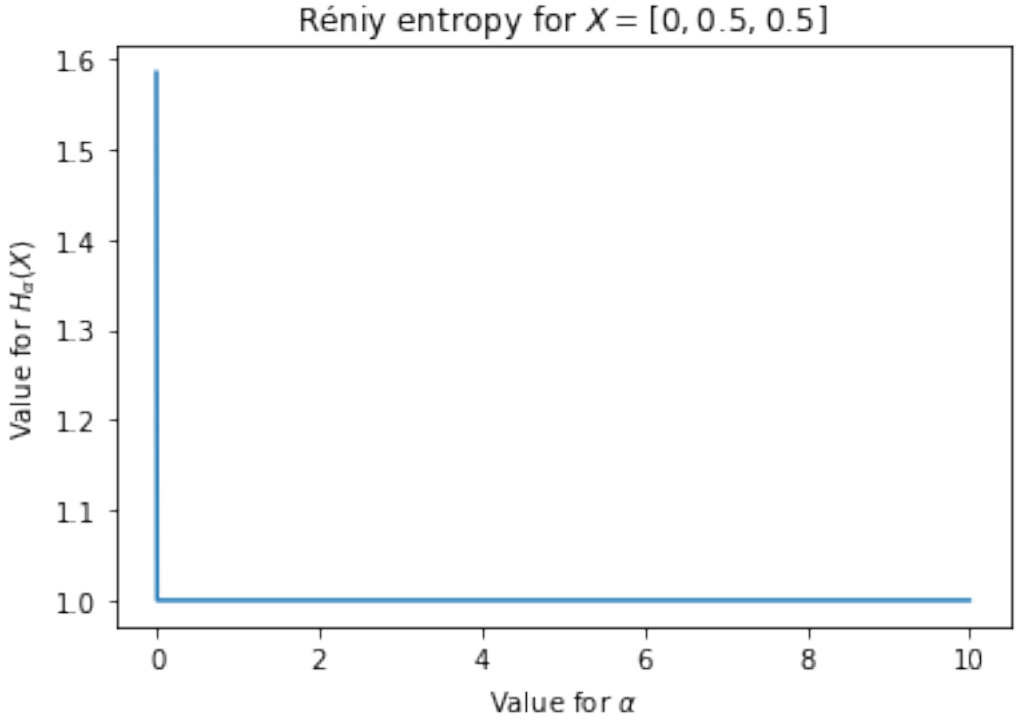
```
In [56]: alphas = np.linspace(0, 10, 1000)

In [59]: renyi_entropy_3(alphas, X1)[:10]

Out[59]: array([1.5849625 , 1.58414417, 1.58332491, 1.58250473, 1.58168363,
                1.58086162, 1.58003871, 1.5792149 , 1.57839021, 1.57756464])

In [71]: def plot_renyi_entropy(alphas, X):
             fig = plt.figure()
             plt.plot(alphas, renyi_entropy_3(alphas, X))
             plt.xlabel(r"Value for $\alpha$")
             plt.ylabel(r"Value for $H_{\alpha}(X)$")
             plt.title(r"Réniy entropy for $X={}$".format(X[:10]))
             plt.show()
             # return fig

In [72]: plot_renyi_entropy(alphas, X1)
```



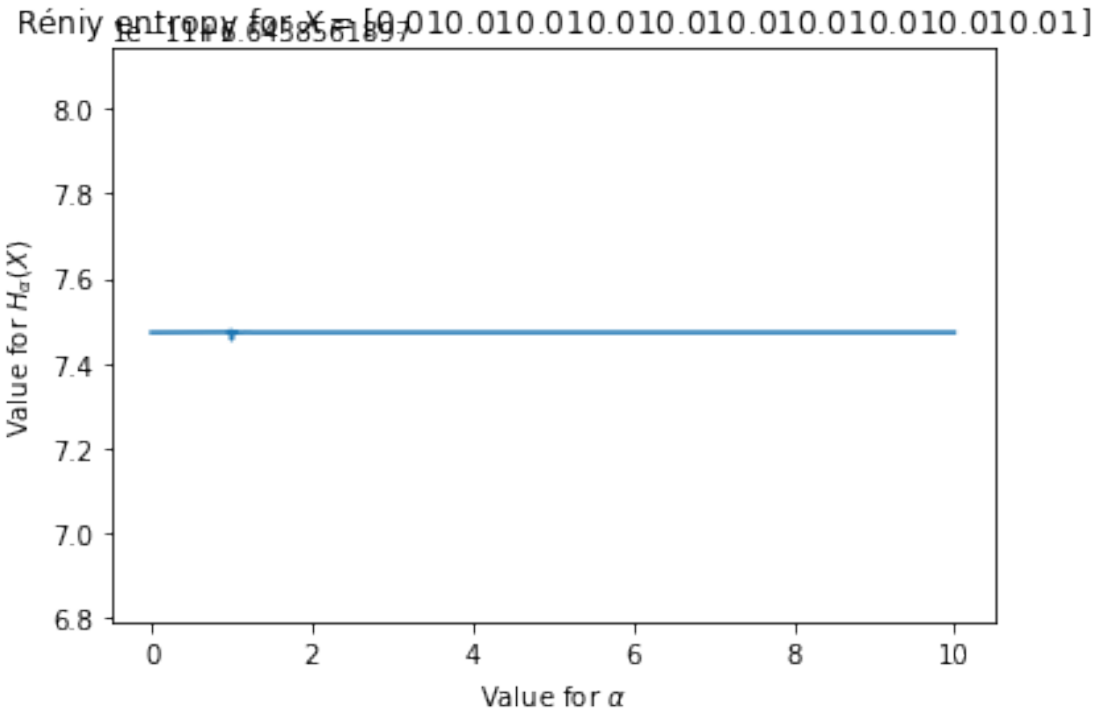

```
In [73]: plot_renyi_entropy(alphas, X2)
```



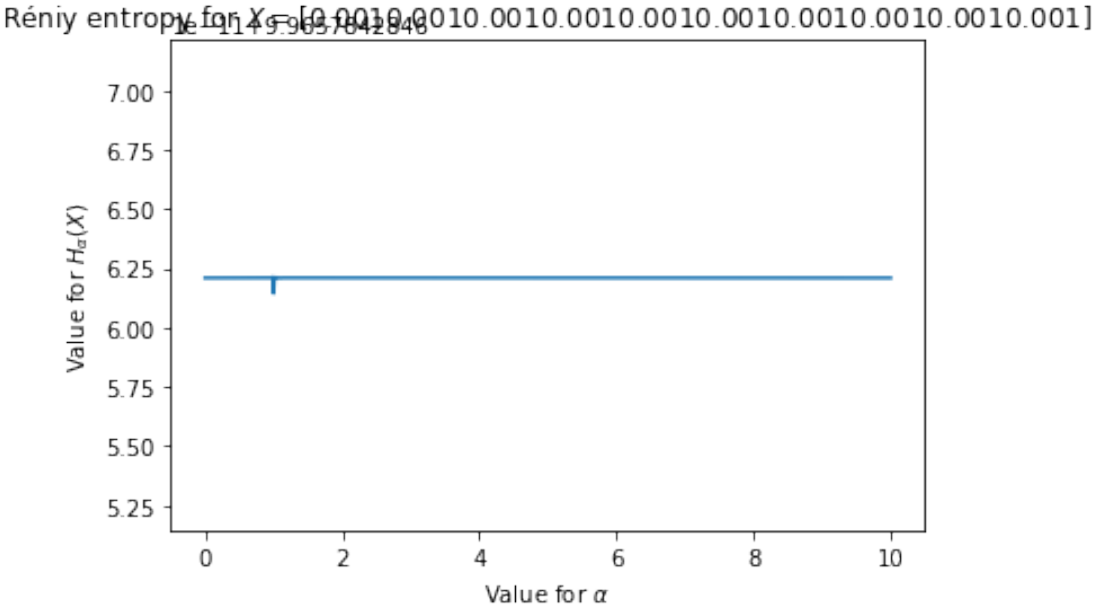
In [74]: `plot_renyi_entropy(alphas, X3)`



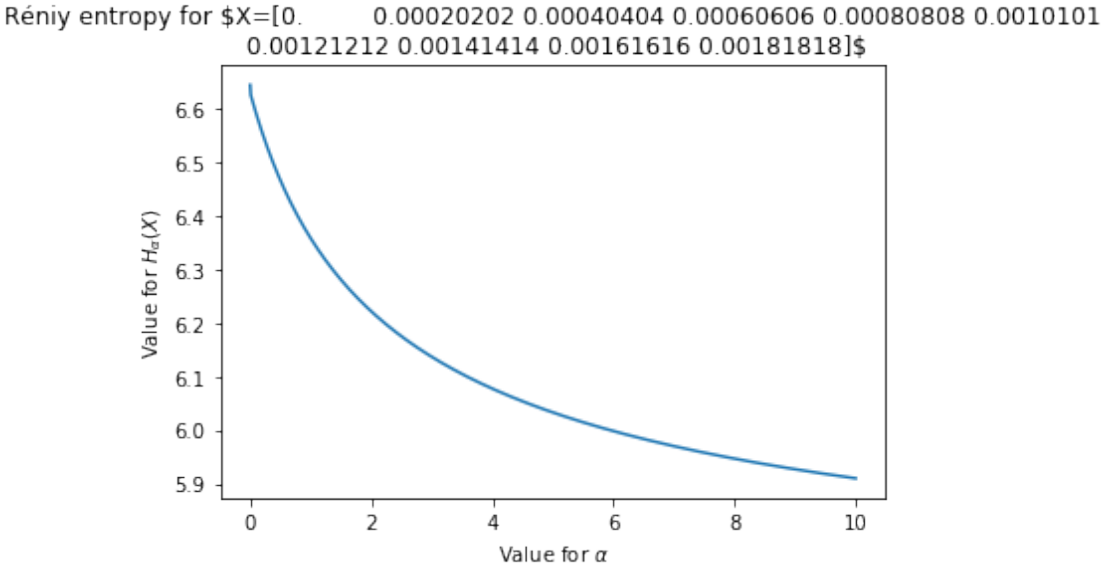
In [75]: plot_renyi_entropy(alphas, X4)



In [76]: plot_renyi_entropy(alphas, X5)



```
In [77]: plot_renyi_entropy(alphas, X6)
```



2.5 Conclusion

It is not surprising that $H_\alpha(X)$ appears to be continuous as a function of α, as one can easily verify that it is.