Manual_implementation_of_the_Mersenne_twister_PseudoRandom_N

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2 Manual implementation of the Mersenne twister PseudoRandom Number Generator (PRNG)

This small notebook is a short experiment, to see if I can implement the Mersenne twister Pseudo-Random Number Generator (PRNG).

And then I want to use it to define a `rand()` function, and use it to samples from the most famous discrete and continuous probability distributions. Random permutations will also be studied.

- *Reference*: Wikipedia, and this book: "Simulation and the Monte-Carlo method", by R.Y.Rubinstein & D.P.Kroese ([Rubinstein & Kroese, 2017]), chapter 2 pages 52-53.
- *Date*: 11 March 2017.
- *Author*: Lilian Besson.
- *License*: MIT Licensed.

2.1 Common API for the PRNG defined here

First, I want to define a simple object-oriented API, in order to write all the examples of PNRG with the same interface.

```
In [131]: import numpy as np
```

```
In [132]: class PRNG(object):
              """Base class for any Pseudo-Random Number Generator."""
              def __init__(self, X0=0):
                  """Create a new PRNG with seed X0."""
                  self.X0 = X0
                  self.X = X0
                  self.t = 0
                  self.max = 0

              def __iter__(self):
                  """self is already an iterator!"""
                  return self

              def seed(self, X0=None):
                  """Reinitialize the current value with X0, or self.X0.

                  - Tip: Manually set the seed if you need reproducibility in your results.
                  """
                  self.t = 0
                  self.X = self.X0 if X0 is None else X0

              def __next__(self):
```

```
        """Produce a next value and return it."""
        # This default PRNG does not produce random numbers!
        self.t += 1
        return self.X

    def randint(self, *args, **kwargs):
        """Return an integer number in [| 0, self.max - 1 |] from the PRNG."""
        return self.__next__()

    def int_samples(self, shape=(1,)):
        """Get a numpy array, filled with integer samples from the PRNG, of shape = 
        # return [ self.randint() for _ in range(size) ]
        return np.fromfunction(np.vectorize(self.randint), shape=shape, dtype=int)

    def rand(self, *args, **kwargs):
        """Return a float number in [0, 1) from the PRNG."""
        return self.randint() / float(1 + self.max)

    def float_samples(self, shape=(1,)):
        """Get a numpy array, filled with float samples from the PRNG, of shape = sh
        # return [ self.rand() for _ in range(size) ]
        return np.fromfunction(np.vectorize(self.rand), shape=shape, dtype=int)
```

2.2 First example: a simple linear congruential generator

Let me start by implementing a simple linear congruential generator, with three parameters m, a, c, defined like this :

- Start from X_0,
- And then follow the recurrence equation:

$$X_{t+1} = (aX_t + c) \mod m.$$

This algorithm produces a sequence $(X_t)_{t \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

In [133]:
```
class LinearCongruentialGenerator(PRNG):
    """A simple linear congruential Pseudo-Random Number Generator."""
    def __init__(self, m, a, c, X0=0):
        """Create a new PRNG with seed X0."""
        super(LinearCongruentialGenerator, self).__init__(X0=X0)
        self.m = self.max = m
        self.a = a
        self.c = c

    def __next__(self):
        """Produce a next value and return it, following the recurrence equation: X_
        self.t += 1
```

```python
            x = self.X
            self.X = (self.a * self.X + self.c) % self.m
            return x
```

The values recommended by the authors, Lewis, Goodman and Miller, are the following:

```python
In [134]: m = 1 << 31 - 1  # 1 << 31 = 2**31
          a = 7 ** 4
          c = 0
```

The seed is important. If $X_0 = 0$, this first example PRNG will only produce $X_t = 0, \forall t$.

```python
In [135]: FirstExample = LinearCongruentialGenerator(m=m, a=a, c=c)
```

```python
In [136]: def test(example, nb=3):
              for t, x in enumerate(example):
                  print("{:>3}th value for {.__class__.__name__} is X_t = {:>10}".format(t, exa
                  if t >= nb - 1:
                      break
```

```python
In [137]: test(FirstExample)
```

```
  0th value for LinearCongruentialGenerator is X_t =          0
  1th value for LinearCongruentialGenerator is X_t =          0
  2th value for LinearCongruentialGenerator is X_t =          0
```

But with any positive seed, the sequence will appear random.

```python
In [138]: SecondExample = LinearCongruentialGenerator(m=m, a=a, c=c, X0=12011993)
```

```python
In [139]: test(SecondExample)
```

```
  0th value for LinearCongruentialGenerator is X_t =   12011993
  1th value for LinearCongruentialGenerator is X_t =  923507769
  2th value for LinearCongruentialGenerator is X_t =   65286809
```

The sequence is completely determined by the seed X_0:

```python
In [140]: SecondExample.seed(12011993)
          test(SecondExample)
```

```
  0th value for LinearCongruentialGenerator is X_t =   12011993
  1th value for LinearCongruentialGenerator is X_t =  923507769
  2th value for LinearCongruentialGenerator is X_t =   65286809
```

Note: I prefer to use this custom class to define iterators, instead of a simple generator (with `yield` keyword) as I want them to have a `.seed(X0)` method.

2.3 Trying to write a cell in cython, for speeding things up

For more details, see this blog post, and this other one.

```
In [141]: # Thanks to https://nbviewer.jupyter.org/gist/minrk/7715212
          from __future__ import print_function
          from IPython.core import page
          def myprint(s):
              try:
                  print(s['text/plain'])
              except (KeyError, TypeError):
                  print(s)
          page.page = myprint
```

```
In [142]: %load_ext cython
```

```
The cython extension is already loaded. To reload it, use:
  %reload_ext cython
```

Then we define a function `LinearCongruentialGenerator_next`, in a Cython cell.

```
In [143]: %%cython
          def nextLCG(int x, int a, int c, int m):
              """Compute x, nextx = (a * x + c) % m, x in Cython."""
              cdef int nextx = (a * x + c) % m
              return (x, nextx)
```

```
In [144]: from __main__ import nextLCG
          nextLCG
          nextLCG?
```

```
Out[144]: <function _cython_magic_dde6682b939b6e9ea0a22da681a4bea1.nextLCG>
```

```
Docstring: Compute x, nextx = (a * x + c) % m, x in Cython.
Type:      builtin_function_or_method
```

Then it's easy to use it to define another Linear Congruential Generator.

```
In [145]: class CythonLinearCongruentialGenerator(LinearCongruentialGenerator):
              """A simple linear congruential Pseudo-Random Number Generator, with Cython acce

              def __next__(self):
                  """Produce a next value and return it, following the recurrence equation: X_
                  self.t += 1
                  x, self.X = nextLCG(self.X, self.a, self.c, self.m)
                  return x
```

Let compare it with the first implementation (using pure Python).

```
In [146]: NotCythonSecondExample = LinearCongruentialGenerator(m=m, a=a, c=c, X0=13032017)
          CythonSecondExample = CythonLinearCongruentialGenerator(m=m, a=a, c=c, X0=13032017)
```

They both give the same values, that's a relief.

```
In [147]: test(NotCythonSecondExample)
          test(CythonSecondExample)
```

```
  0th value for LinearCongruentialGenerator is X_t =   13032017
  1th value for LinearCongruentialGenerator is X_t =  151359921
  2th value for LinearCongruentialGenerator is X_t =  490433809
  0th value for CythonLinearCongruentialGenerator is X_t =   13032017
  1th value for CythonLinearCongruentialGenerator is X_t =  151359921
  2th value for CythonLinearCongruentialGenerator is X_t =  490433809
```

The speedup is not great, but still visible.

```
In [148]: %timeit [ NotCythonSecondExample.randint() for _ in range(1000000) ]
          %timeit [ CythonSecondExample.randint() for _ in range(1000000) ]
```

```
1 loop, best of 3: 766 ms per loop
1 loop, best of 3: 729 ms per loop
```

```
In [149]: %prun min(CythonSecondExample.randint() for _ in range(1000000))
```

```
         4000005 function calls in 1.291 seconds

   Ordered by: internal time

   ncalls  tottime  percall  cumtime  percall filename:lineno(function)
  1000000    0.481    0.000    0.584    0.000 <ipython-input-145-d36c39b118f6>:4(__next__)
  1000001    0.372    0.000    1.184    0.000 <string>:1(<genexpr>)
  1000000    0.228    0.000    0.812    0.000 <ipython-input-132-93560edf79a4>:28(randint)
        1    0.107    0.107    1.291    1.291 {built-in method builtins.min}
  1000000    0.103    0.000    0.103    0.000 {_cython_magic_dde6682b939b6e9ea0a22da681a4bea1.r
        1    0.000    0.000    1.291    1.291 {built-in method builtins.exec}
        1    0.000    0.000    1.291    1.291 <string>:1(<module>)
        1    0.000    0.000    0.000    0.000 {method 'disable' of '_lsprof.Profiler' objects}
```

2.4 Checking and plotting the result?

First, we can generate a matrix of samples, as random floats in $[0, 1)$, and check that the mean is about $1/2$:

```
In [150]: shape = (400, 400)
          image = SecondExample.float_samples(shape)
```

```
In [151]: np.mean(image), np.var(image)
```

```
Out[151]: (0.4996821506033815, 0.0833934768218O3994)
```

What about the speed? Of course, a hand-written Python code will always be really slower than a C-extension code, and the PRNG from the modules `random` or `numpy.random` are written in C (or Cython), and so will always be faster. But how much faster?

```
In [152]: import random
          import numpy.random

          print(np.mean(SecondExample.float_samples(shape)))
          print(np.mean([ [ random.random() for _ in range(shape[0]) ] for _ in range(shape[1]
          print(np.mean(numpy.random.random(shape)))
```

```
0.500181275654
0.500273447015
0.499487700229
```

```
In [153]: %timeit SecondExample.float_samples(shape)
          %timeit [ [ random.random() for _ in range(shape[0]) ] for _ in range(shape[1]) ]
          %timeit numpy.random.random(shape)
```

```
1 loop, best of 3: 271 ms per loop
100 loops, best of 3: 22.2 ms per loop
1000 loops, best of 3: 1.56 ms per loop
```

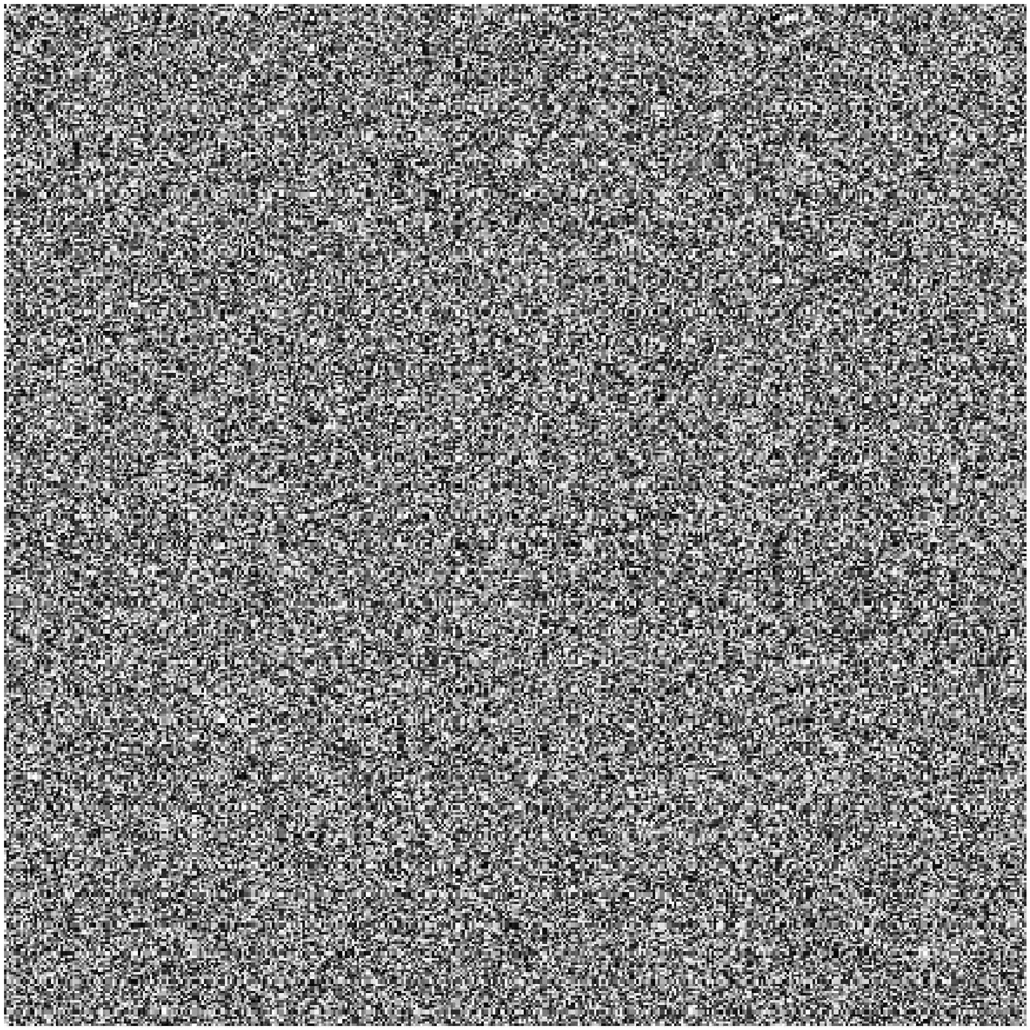
This was expected: of course `numpy.random.` functions are written and optimized to generate thousands of samples quickly, and of course my hand-written Python implementation for `LinearCongruentialGenerator` is slower than the C-code generating the module `random`.

We can also plot this image as a grayscaled image, in order to visualize this "randomness" we just created.

```
In [268]: %matplotlib inline
          import matplotlib.pyplot as plt

          def showimage(image):
              plt.figure(figsize=(8, 8))
              plt.imshow(image, cmap='gray', interpolation='none')
              plt.axis('off')
              plt.show()
```

In [269]: showimage(image)

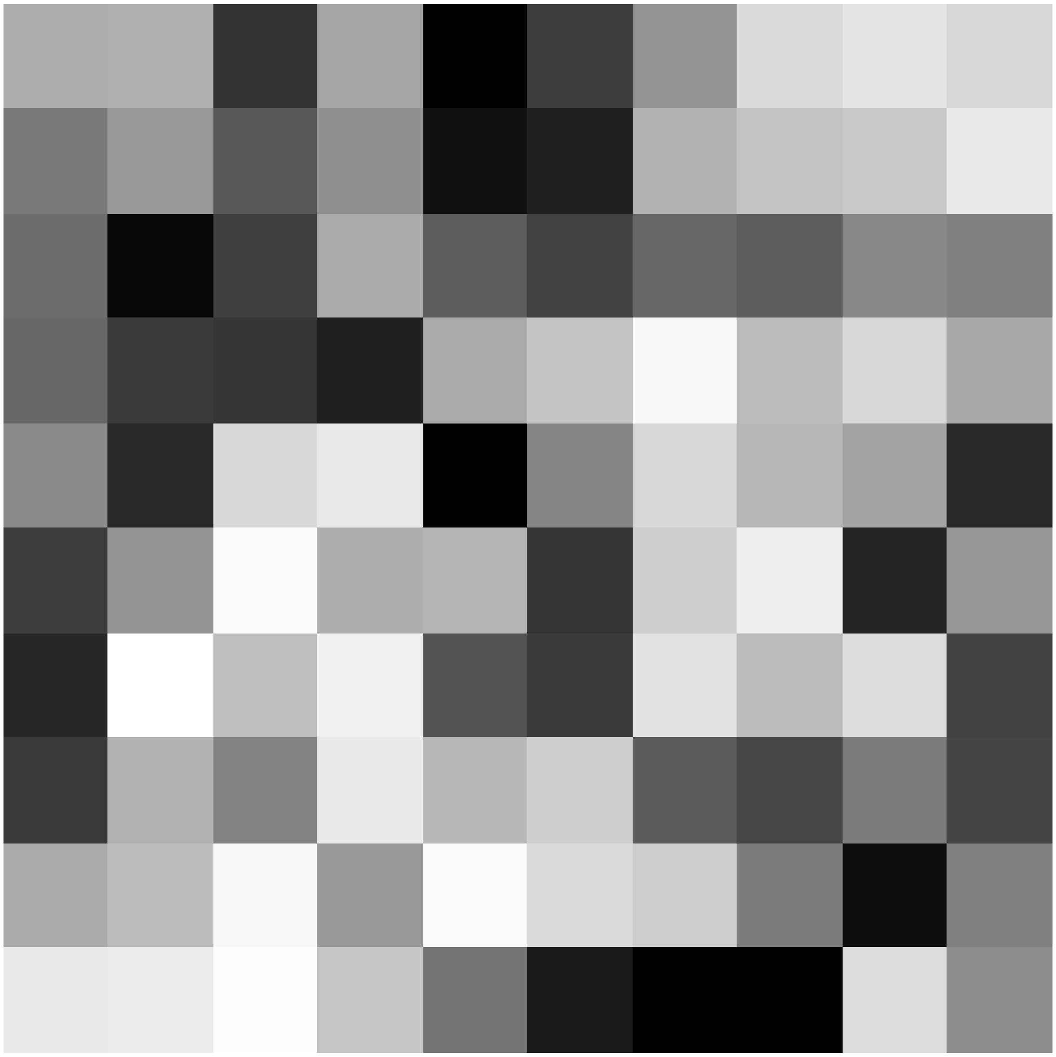
```
showimage(image)
```



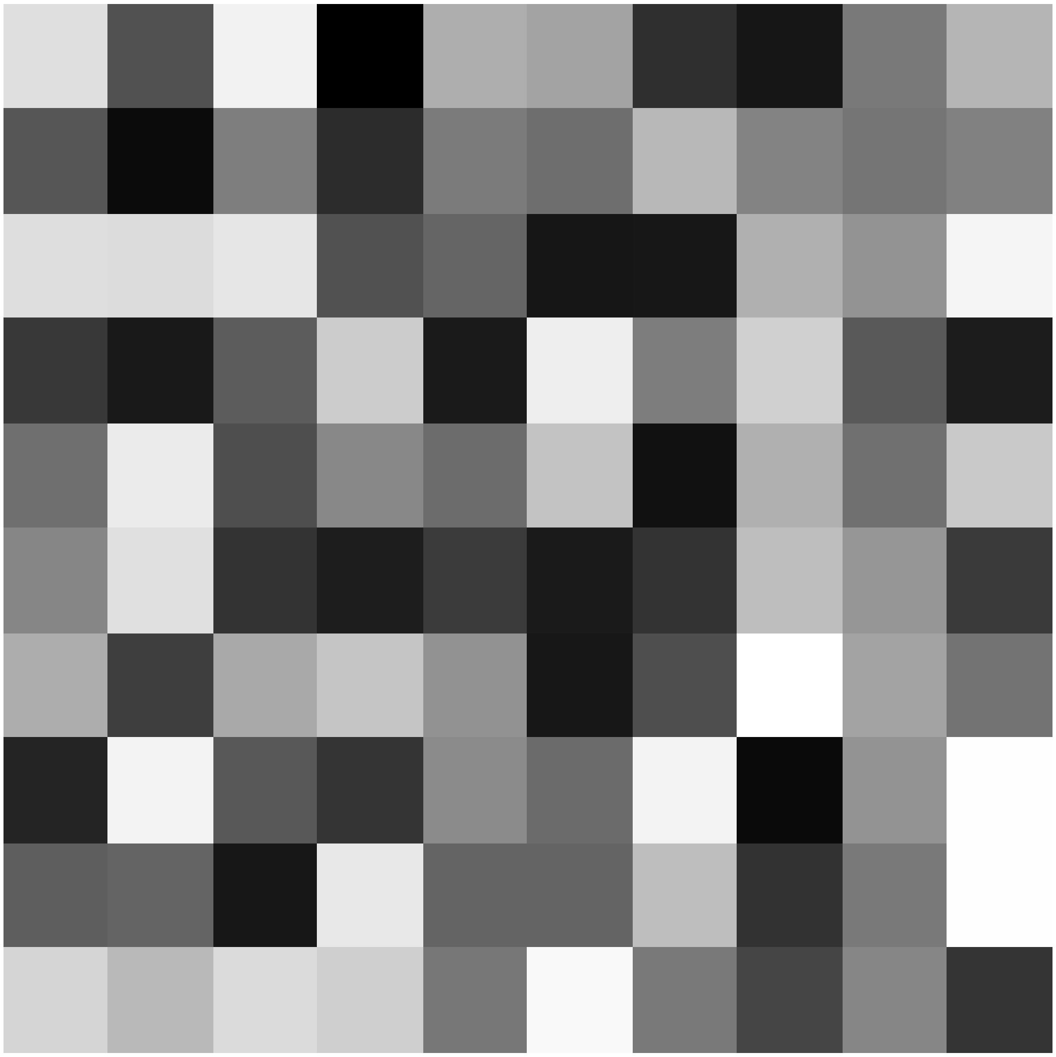
It looks good already! We can't see any recurrence, but we see a regularity, with small squares. And it does not seem to depend too much on the seed:

In [270]:

```
SecondExample.seed(11032017)
image = SecondExample.float_samples((10, 10))
showimage(image)
```

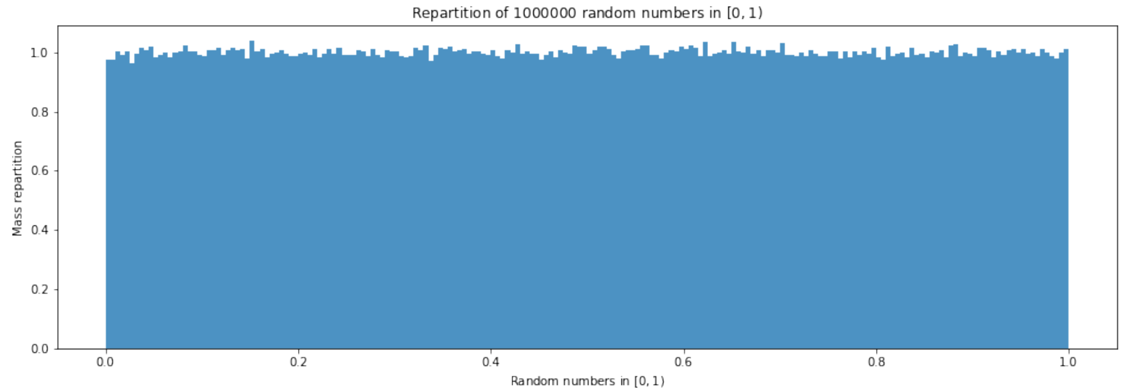


```
In [271]: SecondExample.seed(1103201799)
          image = SecondExample.float_samples((10, 10))
          showimage(image)
```



We can also visualize the generated numbers with a histogram, to visually check that the random numbers in $[0,1)$ are indeed "uniformly" located.

```
In [272]: def plotHistogram(example, nb=100000, bins=200):
              numbers = example.float_samples((nb,))
              plt.figure(figsize=(16, 5))
              plt.hist(numbers, bins=bins, normed=True, alpha=0.8)
              plt.xlabel("Random numbers in $[0, 1)$")
              plt.ylabel("Mass repartition")
              plt.title("Repartition of ${}$ random numbers in $[0, 1)$".format(nb))
              plt.show()

In [273]: plotHistogram(SecondExample, 1000000, 200)
```



2.5 A second example: Multiple-Recursive Generator

Let start by writing a generic Multiple Recursive Generator, which is defined by the following linear recurrence equation, of order $k \geq 1$:

- Start from X_0, with a false initial history of $(X_{-k+1}, X_{-k}, \ldots, X_{-1})$,
- And then follow the recurrence equation:

$$X_t = (a_1 X_{t-1} + \cdots + a_k X_{t-k}) \mod m.$$

This algorithm produces a sequence $(X_t)_{t \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

```
In [160]: class MultipleRecursiveGenerator(PRNG):
              """A Multiple Recursive Pseudo-Random Number Generator (MRG), with one sequence
              def __init__(self, m, a, X0):
                  """Create a new PRNG with seed X0."""
                  assert np.shape(a) == np.shape(X0), "Error: the weight vector a must have the
                  super(MultipleRecursiveGenerator, self).__init__(X0=X0)
                  self.m = self.max = m
                  self.a = a

              def __next__(self):
                  """Produce a next value and return it, following the recurrence equation: X_
                  self.t += 1
                  x = self.X[0]
                  nextx = (np.dot(self.a, self.X)) % self.m
                  self.X[1:] = self.X[:-1]
                  self.X[0] = nextx
                  return x
```

For example, with an arbitrary choice of $k = 3$, of weights $a = [10, 9, 8]$ and $X_0 = [10, 20, 30]$:

```
In [161]: m = (1 << 31) - 1
          X0 = np.array([10, 20, 30])
          a = np.array([10, 9, 8])

          ThirdExample = MultipleRecursiveGenerator(m, a, X0)

          test(ThirdExample)
```

```
  0th value for MultipleRecursiveGenerator is X_t =         10
  1th value for MultipleRecursiveGenerator is X_t =        520
  2th value for MultipleRecursiveGenerator is X_t =       5450
```

We can again check for the mean and the variance of the generated sequence:

```
In [275]: shape = (400, 400)
          image = ThirdExample.float_samples(shape)
          np.mean(image), np.var(image)

Out[275]: (0.49952060992684566, 0.083438346428071117)
```

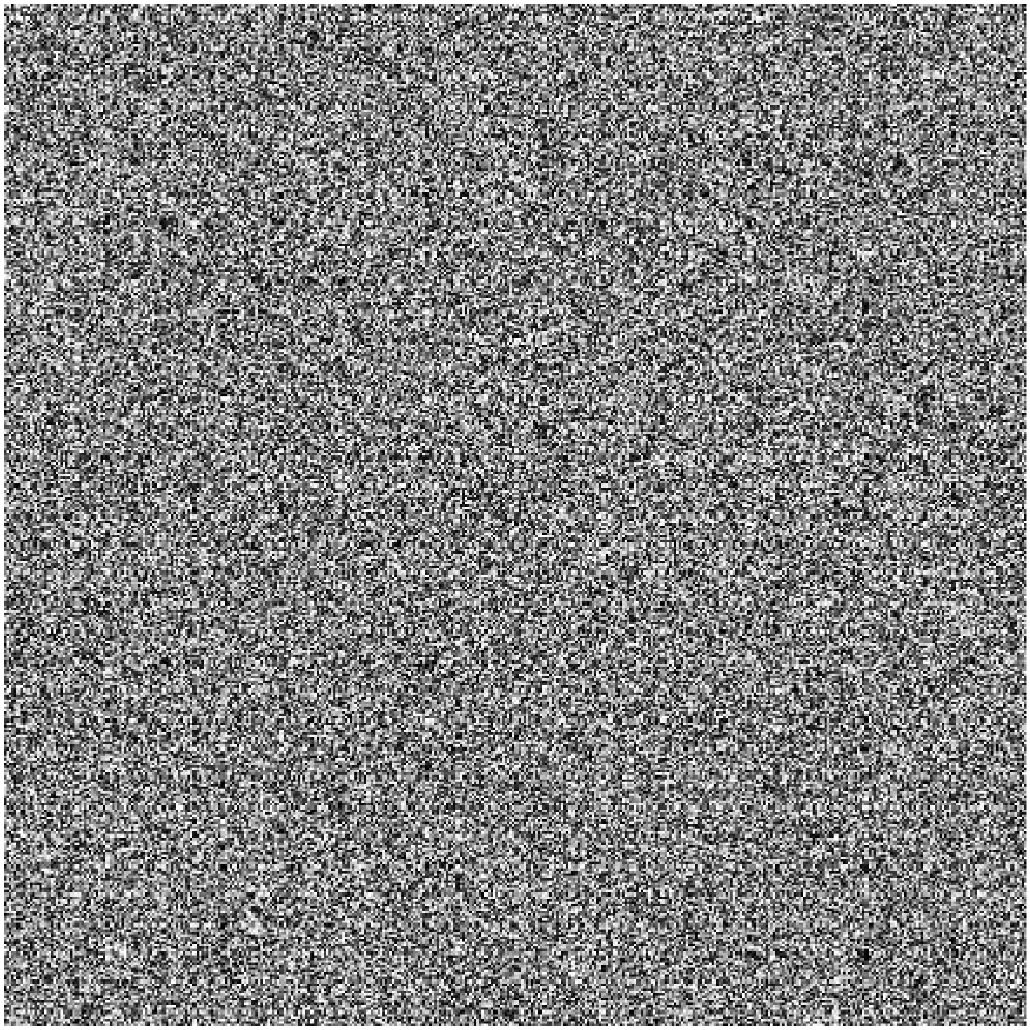
This Multiple Recursive Generator is of course slower than the simple Linear Recurrent Generator:

```
In [163]: %timeit SecondExample.float_samples(shape)
          %timeit ThirdExample.float_samples(shape)
```

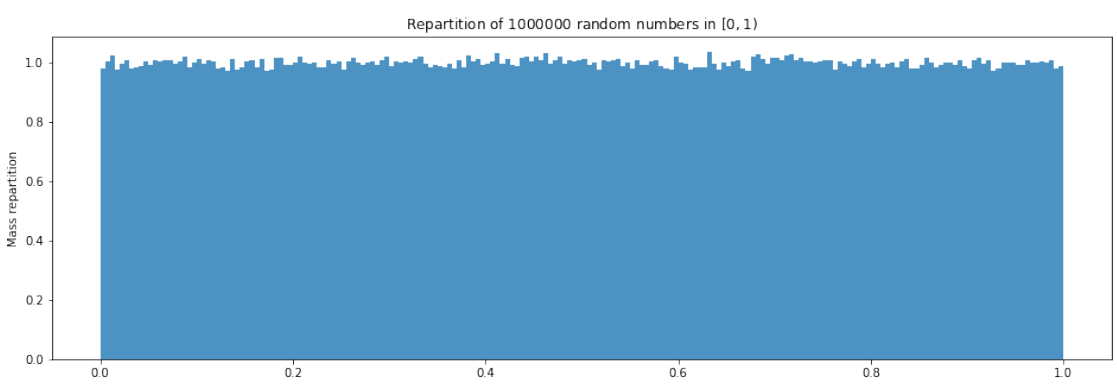
```
1 loop, best of 3: 221 ms per loop
1 loop, best of 3: 895 ms per loop
```

And it seems to work fine as well:

```
In [276]: showimage(image)
```



```
In [277]: plotHistogram(ThirdExample, 1000000, 200)
```



It looks also good!

2.6 A third example: combined Multiple-Recursive Generator, with `MRG32k3a`

Let start by writing a generic Multiple Recursive Generator, which is defined by the following coupled linear recurrence equation, of orders $k_1, k_2 \geq 1$:

- Start from X_0 and Y_0, with a false initial history of $(X_{-k_1+1}, X_{-k_1}, \dots, X_{-1})$ and $(Y_{-k_2+1}, Y_{-k_2}, \dots, Y_{-1})$,
- And then follow the recurrence equation:

$$X_t = (a_1 X_{t-1} + \cdots + a_{k_1} X_{t-k_1}) \mod m.$$

and

$$Y_t = (b_1 Y_{t-1} + \cdots + b_{k_2} Y_{t-k_2}) \mod m.$$

This algorithm produces two sequences $(X_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(X_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, and usually the sequence used for the output is $U_t = X_t - Y_t + \max(m_1, m_2)$.

```
In [166]: class CombinedMultipleRecursiveGenerator(PRNG):
              """A Multiple Recursive Pseudo-Random Number Generator (MRG), with two sequences
              def __init__(self, m1, a, X0, m2, b, Y0):
                  """Create a new PRNG with seeds X0, Y0."""
                  assert np.shape(a) == np.shape(X0), "Error: the weight vector a must have the
                  assert np.shape(b) == np.shape(Y0), "Error: the weight vector b must have the
                  self.t = 0
                  # For X
                  self.m1 = m1
                  self.a = a
                  self.X0 = self.X = X0
                  # For Y
                  self.m2 = m2
                  self.b = b
                  self.Y0 = self.Y = Y0
                  # Maximum integer number produced is max(m1, m2)
                  self.m = self.max = max(m1, m2)

              def __next__(self):
                  """Produce a next value and return it, following the recurrence equation: X_
                  self.t += 1
                  # For X
                  x = self.X[0]
                  nextx = (np.dot(self.a, self.X)) % self.m1
                  self.X[1:] = self.X[:-1]
```

```
                self.X[0] = nextx
                # For Y
                y = self.Y[0]
                nexty = (np.dot(self.b, self.Y)) % self.m2
                self.Y[1:] = self.Y[:-1]
                self.Y[0] = nexty
                # Combine them
                u = x - y + (self.m1 if x <= y else 0)
                return u
```

To obtain the well-known MRG32k3a generator, designed by L'Ecuyer in 1999, we choose these parameters:

```
In [167]: m1 = (1 << 32) - 209                   # important choice!
          a = np.array([0, 1403580, -810728])    # important choice!
          X0 = np.array([1000, 10000, 100000])   # arbitrary choice!

          m2 = (1 << 32) - 22853                 # important choice!
          b = np.array([527612, 0, -1370589])    # important choice!
          Y0 = np.array([5000, 50000, 500000])   # arbitrary choice!

          MRG32k3a = CombinedMultipleRecursiveGenerator(m1, a, X0, m2, b, Y0)

          test(MRG32k3a)
```

```
  0th value for CombinedMultipleRecursiveGenerator is X_t = 4294963087
  1th value for CombinedMultipleRecursiveGenerator is X_t = 1442746955
  2th value for CombinedMultipleRecursiveGenerator is X_t =  970596549
```

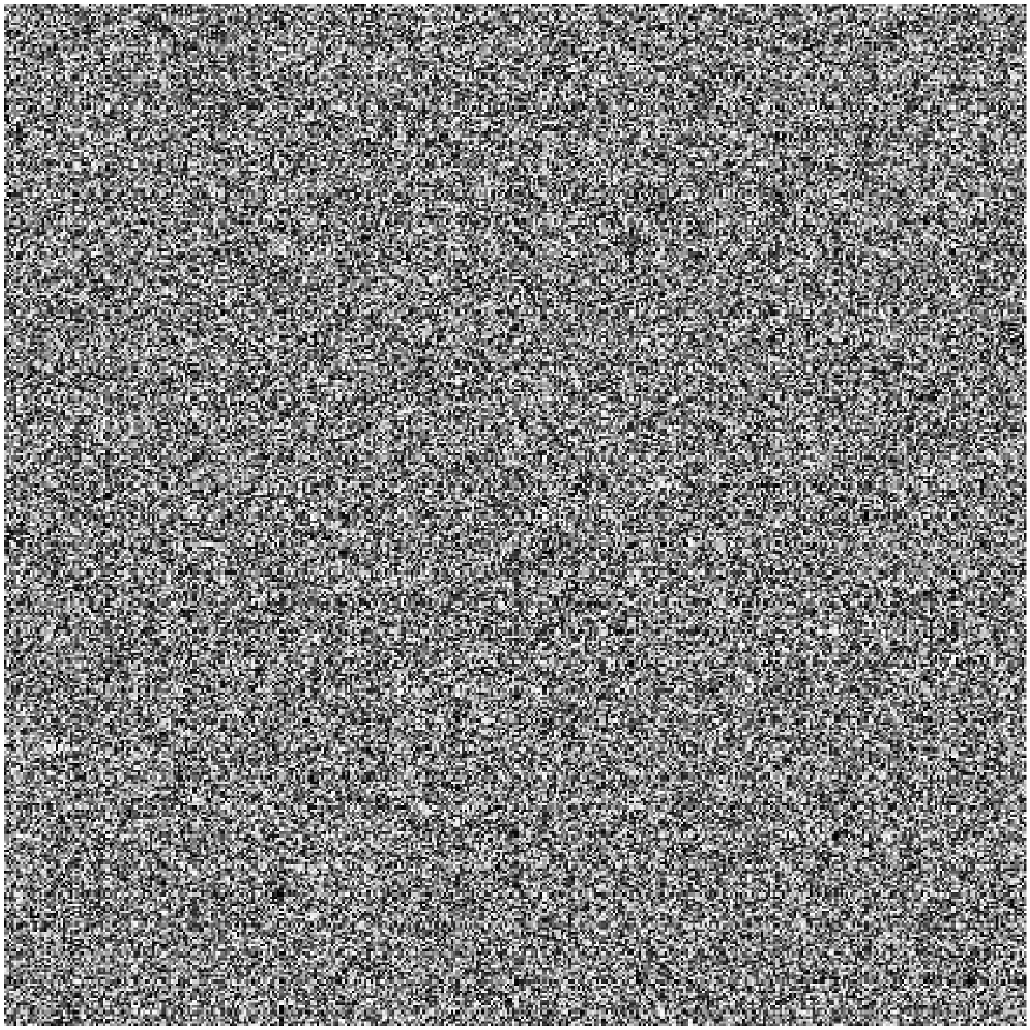
We can again check for the mean and the variance of the generated sequence:

```
In [278]: shape = (400, 400)
          image = MRG32k3a.float_samples(shape)
          np.mean(image), np.var(image)

Out[278]: (0.49952650455296843, 0.08318110283764904)
```

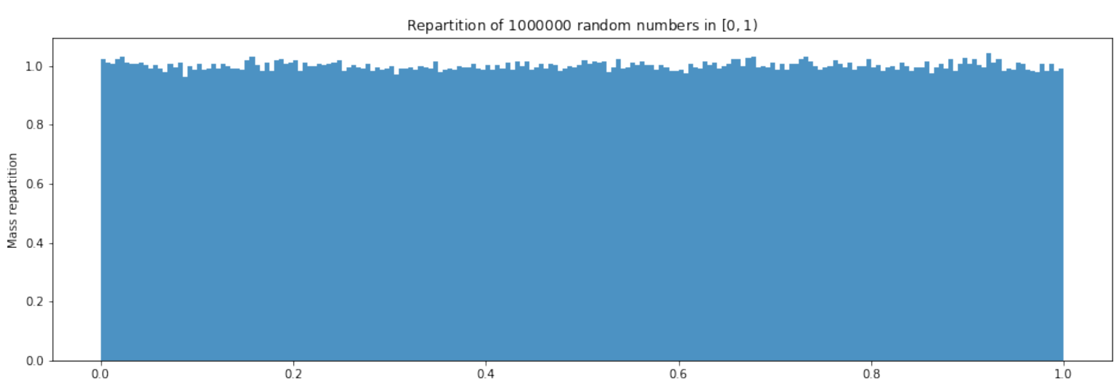
This combined Multiple Recursive Generator is of course slower than the simple Multiple Recursive Generator and the simple Linear Recurrent Generator:

```
In [169]: %timeit SecondExample.float_samples(shape)
          %timeit ThirdExample.float_samples(shape)
          %timeit MRG32k3a.float_samples(shape)

1 loop, best of 3: 251 ms per loop
1 loop, best of 3: 863 ms per loop
1 loop, best of 3: 1.24 s per loop
```

```
In [279]: showimage(image)
```



```
In [280]: plotHistogram(MRG32k3a, 1000000, 200)
```



This one looks fine too!

2.7 Finally, the Mersenne twister PRNG

I won't explain all the details, and will follow closely the notations from my reference book [Rubinstein & Kroese, 2017]. It will be harder to implement!

First, let us compute the period of the PRNG we will implement, with the default values for the parameters $w = 32$ (word length) and $n = 624$ ("big" integer).

2.7.1 Period

```
In [172]: w = 32
          n = 624
```

```
In [173]: def MersenneTwisterPeriod(n, w):
              return (1 << (w * (n - 1) + 1)) - 1

          MersenneTwisterPeriod(n, w) == (2 ** 19937) - 1
```

```
Out[173]: True
```

2.7.2 Random seeds

Then we need to use a previously defined PRNG to set the random seeds.

To try to have "really random" seeds, let me use that classical trick of using the system time as a source of initial randomness.

- Namely, I will use the number of microseconds in the current time stamp as the seed for a `LinearCongruentialGenerator`,
- Then use it to generate the seeds for a `MRG32k3a` generator,
- And finally use it to get the seed for the Mersenne twister.

```
In [174]: from datetime import datetime

          def get_seconds():
              d = datetime.today().timestamp()
              s = 1e6 * (d - int(d))
              return int(s)
```

```
In [175]: get_seconds()   # Example
```

```
Out[175]: 785506
```

In [176]:
```python
def seed_rows(example, n, w):
    return example.int_samples((n,))

def random_Mersenne_seed(n, w):
    linear = LinearCongruentialGenerator(m=(1 << 31) - 1, a=7 ** 4, c=0, X0=get_secon
    assert w == 32, "Error: only w = 32 was implemented"
    m1 = (1 << 32) - 209                        # important choice!
    a = np.array([0, 1403580, -810728])    # important choice!
    X0 = np.array(linear.int_samples((3,)))  # random choice!
    m2 = (1 << 32) - 22853                      # important choice!
    b = np.array([527612, 0, -1370589])    # important choice!
    Y0 = np.array(linear.int_samples((3,)))  # random choice!
    MRG32k3a = CombinedMultipleRecursiveGenerator(m1, a, X0, m2, b, Y0)
    seed = seed_rows(MRG32k3a, n, w)
    assert np.shape(seed) == (n,)
    return seed

example_seed = random_Mersenne_seed(n, w)
example_seed
```

Out[176]:
```
array([1871239779,  613260058,  244547519, 3267481671, 1554624298,
       1961991761, 3811287966, 4176129021,  848956982,  121246666,
       1754035200, 3424467876,  866268922,  230379068, 1178928465,
       2097034094, 1445939073,  398964532,  462460512, 2750298176,
       3359458013, 1075693109, 3633367586, 3584582396, 1524185041,
       3120497617, 1384358948,  760092626,  468632607, 3718611854,
        170312151,   93043999, 1302889854, 1822754143, 3094198579,
        358234833,  563461773, 3236562444, 1564295301, 3728669490,
        323737918, 4138492073, 2493003270,  437271590, 2630117485,
       2097611595, 3253545216, 2375586543, 2334725582, 2516044957,
        294245058, 2177777445, 1309507195, 3435158184, 1695128490,
       1098019007, 1029996593, 3838670278,  372127336,  262477370,
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```
In [177]: for xi in example_seed:
              print("Integer xi = {:>12} and in binary, bin(xi) = {:>34}".format(xi, bin(xi)))
```

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Integer xi =   1871239779 and in binary, bin(xi) =  0b1101111100010001101101001100011
Integer xi =    613260058 and in binary, bin(xi) =   0b100100100011011001101100011010
Integer xi =    244547519 and in binary, bin(xi) =     0b1110100100110111111110111111
Integer xi =   3267481671 and in binary, bin(xi) = 0b11000010110000001110100000100011 1
Integer xi =   1554624298 and in binary, bin(xi) =  0b1011100101010011010111100101010
Integer xi =   1961991761 and in binary, bin(xi) =  0b1110100111100011001111001010001
Integer xi =   3811287966 and in binary, bin(xi) = 0b11100011001010111010001110011110
Integer xi =   4176129021 and in binary, bin(xi) = 0b11111000111010101010101111111101
Integer xi =    848956982 and in binary, bin(xi) =   0b110010100110100000111000110110
Integer xi =    121246666 and in binary, bin(xi) =      0b111001110100001001111001010
Integer xi =   1754035200 and in binary, bin(xi) =  0b1101000100011000111010000000000
Integer xi =   3424467876 and in binary, bin(xi) = 0b11001100000111010011101110100100
Integer xi =    866268922 and in binary, bin(xi) =   0b110011101000100011011011111010
Integer xi =    230379068 and in binary, bin(xi) =     0b1101101110110100111000111100
Integer xi =   1178928465 and in binary, bin(xi) =  0b1000110010001010000010101010001
Integer xi =   2097034094 and in binary, bin(xi) =  0b1111100111111100011001101101110
Integer xi =   1445939073 and in binary, bin(xi) =  0b1010110001011110100011110000001
Integer xi =    398964532 and in binary, bin(xi) =    0b10111110001111011011100110100
Integer xi =    462460512 and in binary, bin(xi) =    0b11011100100001001011001100000
Integer xi =   2750298176 and in binary, bin(xi) = 0b10100011111011100011100001000000
Integer xi =   3359458013 and in binary, bin(xi) = 0b11001000001111010100001011011101
Integer xi =   1075693109 and in binary, bin(xi) =  0b1000000000111011100011000110101
Integer xi =   3633367586 and in binary, bin(xi) = 0b11011000100100001100101000100010
Integer xi =   3584582396 and in binary, bin(xi) = 0b11010101101010000110001011111100
Integer xi =   1524185041 and in binary, bin(xi) =  0b1011010110110010011011111010001
Integer xi =   3120497617 and in binary, bin(xi) = 0b10111001111111110000001111010001
Integer xi =   1384358948 and in binary, bin(xi) =  0b1010010100000111010010000100100
Integer xi =    760092626 and in binary, bin(xi) =   0b101101010011100001011111010010
Integer xi =    468632607 and in binary, bin(xi) =    0b11011111011101100010000011111
Integer xi =   3718611854 and in binary, bin(xi) = 0b11011101101001011000001110001110
Integer xi =    170312151 and in binary, bin(xi) =     0b1010001001101100000111010111
Integer xi =     93043999 and in binary, bin(xi) =      0b101100010111011110100011111
Integer xi =   1302889854 and in binary, bin(xi) =  0b1001101101010001000010101111110
Integer xi =   1822754143 and in binary, bin(xi) =  0b1101100101001010000010101011111
Integer xi =   3094198579 and in binary, bin(xi) = 0b10111000011011011011100100110011
Integer xi =    358234833 and in binary, bin(xi) =    0b10101010110100011101011010001
Integer xi =    563461773 and in binary, bin(xi) =   0b100001100101011011111010001101
Integer xi =   3236562444 and in binary, bin(xi) = 0b11000000111010100000011000001100
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Integer xi =   2493003270 and in binary, bin(xi) = 0b10010100100110000011011000000110
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Integer xi =    294245058 and in binary, bin(xi) =    0b10001100010011101001011000010
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Integer xi =   1309507195 and in binary, bin(xi) =  0b1001110000011010111111001111011
Integer xi =   3435158184 and in binary, bin(xi) = 0b11001100110000000101101010101000
Integer xi =   1695128490 and in binary, bin(xi) =  0b1100101000010011001101110101010
Integer xi =   1098019007 and in binary, bin(xi) =  0b1000001011100100111000010111111
Integer xi =   1029996593 and in binary, bin(xi) =   0b111101011001001000000000110001
Integer xi =   3838670278 and in binary, bin(xi) = 0b11100100110011010111010111000110
Integer xi =    372127336 and in binary, bin(xi) =    0b10110001011100011011001101000
Integer xi =    262477370 and in binary, bin(xi) =     0b1111101001010001011000111010
Integer xi =   3888725742 and in binary, bin(xi) = 0b11100111110010010011111011101110
Integer xi =    138313317 and in binary, bin(xi) =     0b1000001111100111111001100101
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Integer xi =    761864309 and in binary, bin(xi) =   0b101101011010010010000001110101
Integer xi =   3849143015 and in binary, bin(xi) = 0b11100101011011010100001011100111
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Integer xi =    638977987 and in binary, bin(xi) =   0b100110000101100000011111000011
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Integer xi =    459167486 and in binary, bin(xi) =    0b11011010111100101011011111110
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Integer xi =   4281399551 and in binary, bin(xi) = 0b11111111001100001111100011111111
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Integer xi =   3973263892 and in binary, bin(xi) = 0b11101100110100110011001000010100
Integer xi =   1197689553 and in binary, bin(xi) =  0b1000111011000110100101011010001
Integer xi =   2927586334 and in binary, bin(xi) = 0b10101110011111110110110000011110
Integer xi =   3652428993 and in binary, bin(xi) = 0b11011001101100111010010011000001
Integer xi =   1726975315 and in binary, bin(xi) =  0b1100110111011111000110101010011
Integer xi =   1410287868 and in binary, bin(xi) =  0b1010100000011110100100011111100
Integer xi =   2788568833 and in binary, bin(xi) = 0b10100110001101100010111100000001
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Integer xi =   2628719687 and in binary, bin(xi) = 0b10011100101011110001010001000111
Integer xi =   1518108225 and in binary, bin(xi) =  0b1011010011111000111111001000001
Integer xi =   2051151561 and in binary, bin(xi) =  0b1111010010000100001011011001001
Integer xi =   4093100836 and in binary, bin(xi) = 0b11110011111101111100001100100100
Integer xi =   3218627959 and in binary, bin(xi) = 0b10111111101100001011101011101111
Integer xi =    866978907 and in binary, bin(xi) =   0b110011101011010000110001011011
Integer xi =      7673933 and in binary, bin(xi) =          0b11101010001100001001101
Integer xi =   1741131969 and in binary, bin(xi) =  0b1100111110001111001000011000001
Integer xi =   1023486388 and in binary, bin(xi) =   0b111101000000010010100110110100
Integer xi =    979415369 and in binary, bin(xi) =   0b111010011000001011000101001001
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Integer xi =    906919976 and in binary, bin(xi) =   0b110110000011101000000000101000
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Integer xi =   2595271731 and in binary, bin(xi) = 0b10011010101100001011010000110011
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Integer xi =    461770900 and in binary, bin(xi) =    0b11011100001100001000010010100
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Integer xi =   1826940947 and in binary, bin(xi) =  0b1101100111001001110100000010011
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Integer xi =   2944579553 and in binary, bin(xi) = 0b10101111100000101011011111100001
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Integer xi =   2795807192 and in binary, bin(xi) = 0b10100110101001001010000111011000
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Integer xi =   1714589414 and in binary, bin(xi) =  0b1100110001100101000111011100110
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Integer xi = 1931256725 and in binary, bin(xi) =  0b1110011000111001010001110010101
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Integer xi =   3740176171 and in binary, bin(xi) = 0b11011110111011101000111100101011
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Integer xi =  3006378686 and in binary, bin(xi) = 0b10110011001100011011001010111110
Integer xi =   232795670 and in binary, bin(xi) =     0b1101111000000010111000010110
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Integer xi =   315054746 and in binary, bin(xi) =    0b10010110001110101101010011010
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Integer xi =  4115602201 and in binary, bin(xi) = 0b11110101010011110001101100011001
Integer xi =   417078816 and in binary, bin(xi) =    0b11000110111000001111000100000
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Integer xi =  4219712046 and in binary, bin(xi) = 0b11111011100000111011001000101110
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Integer xi =  3232561385 and in binary, bin(xi) = 0b11000000101011001111100011101001
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Integer xi =  3570148724 and in binary, bin(xi) = 0b11010100110011000010010101110100
Integer xi =  3326938343 and in binary, bin(xi) = 0b11000110010011010000110011100111
Integer xi =  4105697636 and in binary, bin(xi) = 0b11110100101101111111100101100100
Integer xi =  2632643363 and in binary, bin(xi) = 0b10011100111010101111001100100011
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Integer xi =   1440099044 and in binary, bin(xi) =  0b1010101110101100010101011100100
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Integer xi =   2378600510 and in binary, bin(xi) = 0b10001101110001101001000000111110
Integer xi =     13651577 and in binary, bin(xi) =         0b110100000100111001111001
Integer xi =   3813736373 and in binary, bin(xi) = 0b11100011010100001111111110110101
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Integer xi =   2287913525 and in binary, bin(xi) = 0b10001000010111101100101000110101
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Integer xi =     23444368 and in binary, bin(xi) =        0b1011001011011101110010000
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Integer xi =   2689526255 and in binary, bin(xi) = 0b10100000010011101110100111101111
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Integer xi =    155417897 and in binary, bin(xi) =      0b1001010000110111110100101001
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Integer xi =   3753270291 and in binary, bin(xi) = 0b11011111101101100101110000010011
Integer xi =    287608410 and in binary, bin(xi) =    0b10001001001001000111001011010
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Integer xi =   2292170038 and in binary, bin(xi) = 0b10001000100111111011110100110110
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Integer xi =   2048092710 and in binary, bin(xi) =  0b1111010000100110110101000100110
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Integer xi =   2693059081 and in binary, bin(xi) = 0b10100000100001001101001000001001
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Integer xi =   3315195164 and in binary, bin(xi) = 0b11000101100110011101110100011100
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Integer xi =   2776544714 and in binary, bin(xi) = 0b10100101011111101011010111001010
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Integer xi =   2512356848 and in binary, bin(xi) = 0b10010101101111111000010111110000
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Integer xi =   1334387131 and in binary, bin(xi) = 0b1001111100010010010000110111011
Integer xi =    697147504 and in binary, bin(xi) =  0b101001100011011010000001110000
Integer xi =   1142438326 and in binary, bin(xi) = 0b1000100000110000011100110110110
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Integer xi =   1351058597 and in binary, bin(xi) = 0b1010000100001111000010010100101
Integer xi =    978466409 and in binary, bin(xi) =  0b111010010100100011011001101001
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Integer xi =    933906235 and in binary, bin(xi) =  0b110111101010100100011100111011
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Integer xi =   3631298850 and in binary, bin(xi) = 0b11011000011100010011100100100010
Integer xi =    643958142 and in binary, bin(xi) =  0b100110011000100000010101111110
Integer xi =    911311576 and in binary, bin(xi) =  0b110110010100011000001011011000
Integer xi =   2529950011 and in binary, bin(xi) = 0b10010110110010111111100100111011
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Integer xi =   3405919962 and in binary, bin(xi) = 0b11001011000000100011011011011010
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Integer xi =    949522371 and in binary, bin(xi) =   0b111000100110001000111111000011
Integer xi =    908076311 and in binary, bin(xi) =   0b110110001000000010010100010111
Integer xi =   1914623417 and in binary, bin(xi) =  0b1110010000111101101010110111001
Integer xi =   1965182279 and in binary, bin(xi) =  0b1110101001000100100110101000111
Integer xi =   2192742552 and in binary, bin(xi) = 0b10000010101100101001100010011000
Integer xi =   3885699423 and in binary, bin(xi) = 0b11100111100110110001000101011111
Integer xi =   1944076290 and in binary, bin(xi) =  0b1110011111000000100000000000010
Integer xi =    541678656 and in binary, bin(xi) =   0b100000010010010101110001000000
Integer xi =   2391536087 and in binary, bin(xi) = 0b10001110100010111111000111010111
Integer xi =    316854319 and in binary, bin(xi) =    0b10010111000101101000000101111
Integer xi =     18424049 and in binary, bin(xi) =        0b1000110010010000011110001
Integer xi =   2485161495 and in binary, bin(xi) = 0b10010100001000001000111000010111
Integer xi =    633174594 and in binary, bin(xi) =   0b100101101111010111101001000010
Integer xi =   3856720693 and in binary, bin(xi) = 0b11100101111000001110001100110101
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Integer xi =   1061663071 and in binary, bin(xi) =   0b111111010001111011000101011111
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Integer xi =   2917247060 and in binary, bin(xi) = 0b10101101111000011010100001010100
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Integer xi =   2946801748 and in binary, bin(xi) = 0b10101111101001001010000001010100
Integer xi =    585112101 and in binary, bin(xi) =   0b100010111000000001101000100101
Integer xi =   2935817224 and in binary, bin(xi) = 0b10101110111111010000010000001000
Integer xi =   2142413707 and in binary, bin(xi) =  0b1111111101100101010001110001011
Integer xi =    398795855 and in binary, bin(xi) =    0b10111110001010010010001001111
Integer xi =   3755006652 and in binary, bin(xi) = 0b11011111110100001101101010111100
Integer xi =   1567043098 and in binary, bin(xi) =  0b1011101011001110010111000011010
Integer xi =    277852780 and in binary, bin(xi) =    0b10000100011111011001001101100
Integer xi =   1675396547 and in binary, bin(xi) =  0b1100011110111001000010111000011
Integer xi =   3583522452 and in binary, bin(xi) = 0b11010101100110000011011010010100
Integer xi =   4265050137 and in binary, bin(xi) = 0b11111110001101111000000000011001
Integer xi =   1174975985 and in binary, bin(xi) =  0b1000110000010001011010111110001
Integer xi =   3348363442 and in binary, bin(xi) = 0b11000111100100111111100010110010
Integer xi =   3541378862 and in binary, bin(xi) = 0b11010011000101010010011100101110
Integer xi =   2730613383 and in binary, bin(xi) = 0b10100010110000001110110101000111
Integer xi =   4240573901 and in binary, bin(xi) = 0b11111100110000100000010111001101
Integer xi =   1064098827 and in binary, bin(xi) =   0b111111011011001101110000001011
Integer xi =   3966639723 and in binary, bin(xi) = 0b11101100011011100001111001101011
Integer xi =   2775717648 and in binary, bin(xi) = 0b10100101011100100001011100010000
Integer xi =   3339443893 and in binary, bin(xi) = 0b11000111000010111101111010110101
Integer xi =   2941191519 and in binary, bin(xi) = 0b10101111010011110000010101011111
Integer xi =   1353655096 and in binary, bin(xi) =  0b1010000101011110010001100111000
Integer xi =   3254930476 and in binary, bin(xi) = 0b11000010000000100100110000101100
Integer xi =    904857047 and in binary, bin(xi) =   0b110101111011110000010111010111
```

```
Integer xi =   3907208435 and in binary, bin(xi) = 0b11101000111000110100010011110011
Integer xi =   2948098651 and in binary, bin(xi) = 0b10101111101110000110101001011011
Integer xi =   1078468758 and in binary, bin(xi) =  0b1000000010010000010000010010110
Integer xi =   3981043271 and in binary, bin(xi) = 0b11101101010010011110011001000111
Integer xi =   3907076547 and in binary, bin(xi) = 0b11101000111000010100000111000011
```

2.7.3 Implementing the Mersenne twister algorithm

Finally, the Mersenne twister can be implemented like this:

```
In [179]: class MersenneTwister(PRNG):
              """The Mersenne twister Pseudo-Random Number Generator (MRG)."""
              def __init__(self, seed=None,
                           w=32, n=624, m=397, r=31,
                           a=0x9908B0DF, b=0x9D2C5680, c=0xEFC60000,
                           u=11, s=7, v=15, l=18):
                  """Create a new Mersenne twister PRNG with this seed."""
                  self.t = 0
                  # Parameters
                  self.w = w
                  self.n = n
                  self.m = m
                  self.r = r
                  self.a = a
                  self.b = b
                  self.c = c
                  self.u = u
                  self.s = s
                  self.v = v
                  self.l = l
                  # For X
                  if seed is None:
                      seed = random_Mersenne_seed(n, w)
                  self.X0 = seed
                  self.X = np.copy(seed)
                  # Maximum integer number produced is 2**w - 1
                  self.max = (1 << w) - 1

              def __next__(self):
                  """Produce a next value and return it, following the Mersenne twister algori
                  self.t += 1
                  # 1. --- Compute x_{t+n}
                  # 1.1.a. First r bits of x_t : left = (x_t >> (w - r)) << (w - r)
                  # 1.1.b. Last w - r bits of x_{t+1} : right = x & ((1 << (w - r)) - 1)
                  # 1.1.c. Concatenate them together in a binary vector x : x = left + right
                  left = self.X[0] >> (self.w - self.r)
                  right = (self.X[1] & ((1 << (self.w - self.r)) - 1))
```

```
        x = (left << (self.w - self.r)) + right
        xw = x % 2              # 1.2. get xw
        if xw == 0:
            xtilde = (x >> 1)            # if xw = 0, xtilde = (x >> 1)
        else:
            xtilde = (x >> 1) ^ self.a   # if xw = 1, xtilde = (x >> 1)   a
        nextx = self.X[self.m] ^ xtilde  # 1.3. x_{t+n} = x_{t+m}   \tilde{x}
        # 2. --- Shift the content of the n rows
        oldx0 = self.X[0]           # 2.a. First, forget x0
        self.X[:-1] = self.X[1:]    # 2.b. shift one index on the left, x1..xn-1 to x
        self.X[-1]  = nextx         # 2.c. write new xn-1
        # 3. --- Then use it to compute the answer, y
        y = nextx                       # 3.a. y = x_{t+n}
        y ^= (y >> self.u)              # 3.b. y = y  (y >> u)
        y ^= ((y << self.s) & self.b)   # 3.c. y = y  ((y << s) & b)
        y ^= ((y << self.v) & self.c)   # 3.d. y = y  ((y << v) & c)
        y ^= (y >> self.l)              # 3.e. y = y  (y >> l)
        return y
```

2.7.4 Small review of bitwise operations

The Python documentation explains how to use bitwise operations easily, and also this page and this StackOverflow answer.

The only difficult part of the algorithm is the first step, when we need to take the first r bits of $X_t =$ `X[0]`, and the last $w - r$ bits of $X_{t+1} =$ `X[1]`. On some small examples, let quickly check that I implemented this correctly:

```
In [180]: def testsplit(x, r=None, w=None):
              if w is None:
                  w = x.bit_length()
              if r is None:
                  r = w - 1
              assert x.bit_length() == w
              left = x >> (w - r)
              right = x % 2 if w == 1 else x & ((1 << (w-r) - 1))
              x2 = (left << (w - r)) + right
              assert x == x2
              print("x = {:10} -> left r={} = {:10} and right w-r={} = {:4} -> x2 = {:10}".forn

          x = 0b10011010
          testsplit(x)
          x = 0b10010011
          testsplit(x)
          x = 0b10011111
          testsplit(x)
          x = 0b11110001
          testsplit(x)
          x = 0b00110001
```

```
testsplit(x)
```

```
x = 0b10011010 -> left r=7 = 0b1001101  and right w-r=1 = 0b0  -> x2 = 0b10011010
x = 0b10010011 -> left r=7 = 0b1001001  and right w-r=1 = 0b1  -> x2 = 0b10010011
x = 0b10011111 -> left r=7 = 0b1001111  and right w-r=1 = 0b1  -> x2 = 0b10011111
x = 0b11110001 -> left r=7 = 0b1111000  and right w-r=1 = 0b1  -> x2 = 0b11110001
x = 0b110001   -> left r=5 = 0b11000    and right w-r=1 = 0b1  -> x2 = 0b110001
```

2.7.5 Mersenne twister algorithm in cython

As for the first example, let us write a Cython function to (try to) compute the next numbers more easily.

My reference was this page of the Cython documentation.

```
In [262]: %%cython
          from __future__ import division
          import cython
          import numpy as np
          # "cimport" is used to import special compile-time information
          # about the numpy module (this is stored in a file numpy.pxd which is
          # currently part of the Cython distribution).
          cimport numpy as np

          # We now need to fix a datatype for our arrays. I've used the variable
          # DTYPE for this, which is assigned to the usual NumPy runtime
          # type info object.
          DTYPE = np.int64
          # "ctypedef" assigns a corresponding compile-time type to DTYPE_t. For
          # every type in the numpy module there's a corresponding compile-time
          # type with a _t-suffix.
          ctypedef np.int64_t DTYPE_t


          @cython.boundscheck(False) # turn off bounds-checking for entire function
          def nextMersenneTwister(np.ndarray[DTYPE_t, ndim=1] X, unsigned long w, unsigned long
              """Produce a next value and return it, following the Mersenne twister algorithm,
              assert X.dtype == DTYPE
              # 1. --- Compute x_{t+n}
              # 1.1.a. First r bits of x_t : left = (x_t >> (w - r)) << (w - r)
              # 1.1.b. Last w - r bits of x_{t+1} : right = x & ((1 << (w - r)) - 1)
              # 1.1.c. Concatenate them together in a binary vector x : x = left + right
              cdef unsigned long x = ((X[0] >> (w - r)) << (w - r)) + (X[1] & ((1 << (w - r)) -
              cdef unsigned long xtilde = 0
              if x % 2 == 0:  # 1.2. get xw
                  xtilde = (x >> 1)            # if xw = 0, xtilde = (x >> 1)
              else:
                  xtilde = (x >> 1) ^ a   # if xw = 1, xtilde = (x >> 1)  a
```

```
        cdef unsigned long nextx = X[m] ^ xtilde  # 1.3. x_{t+n} = x_{t+m}  \tilde{x}
        # 2. --- Shift the content of the n rows
        # oldx0 = X[0]            # 2.a. First, forget x0
        X[:-1] = X[1:]            # 2.b. shift one index on the left, x1..xn-1 to x0..xn
        X[-1]  = nextx            # 2.c. write new xn-1
        # 3. --- Then use it to compute the answer, y
        cdef unsigned long y = nextx        # 3.a. y = x_{t+n}
        y ^= (y >> u)             # 3.b. y = y  (y >> u)
        y ^= ((y << s) & b)       # 3.c. y = y  ((y << s) & b)
        y ^= ((y << v) & c)       # 3.d. y = y  ((y << v) & c)
        y ^= (y >> l)             # 3.e. y = y  (y >> l)
        return y
```

In [263]:
```
nextMersenneTwister
nextMersenneTwister?
```

Out[263]: `<function _cython_magic_254860e6a9a04121ac2ee81a80b68b1a.nextMersenneTwister>`

```
Docstring: Produce a next value and return it, following the Mersenne twister algorithm, implem
Type:      builtin_function_or_method
```

That should be enough to define a Cython version of our `MersenneTwister` class.

In [264]:
```
class CythonMersenneTwister(MersenneTwister):
    """The Mersenne twister Pseudo-Random Number Generator (MRG), accelerated with Cy

    def __next__(self):
        """Produce a next value and return it, following the Mersenne twister algori
        self.t += 1
        return nextMersenneTwister(self.X, self.w, self.m, self.r, self.a, self.u, se
```

2.7.6 Testing our implementations

In [265]:
```
ForthExample = MersenneTwister(seed=example_seed)
CythonForthExample = CythonMersenneTwister(seed=example_seed)
```

In [266]:
```
ForthExample.int_samples((10,))
CythonForthExample.int_samples((10,))
```

Out[266]:
```
array([ 306676890, 2556225439, 1979801398,   27614268, 1499672499,
        1289742183, 3706951411, 3441754698, 3090564280, 1104599484])
```

Out[266]:
```
array([ 306676890, 2556225439, 1979801398,   27614268, 1499672499,
        1289742183, 3706951411, 3441754698, 3090564280, 1104599484])
```

Which one is the quickest?

In [267]:
```
%timeit [ ForthExample.randint() for _ in range(100000) ]
%timeit [ CythonForthExample.randint() for _ in range(100000) ]
```

```
1 loop, best of 3: 402 ms per loop
10 loops, best of 3: 199 ms per loop
```

Using Cython gives only a speedup of $2\times$, that's disappointing!

```
In [187]: %prun [ ForthExample.randint() for _ in range(1000000) ]
```

```
         2000004 function calls in 6.187 seconds

   Ordered by: internal time

   ncalls  tottime  percall  cumtime  percall filename:lineno(function)
  1000000    5.304    0.000    5.304    0.000 <ipython-input-179-740d1415e4c6>:29(__next__)
        1    0.503    0.503    6.179    6.179 <string>:1(<listcomp>)
  1000000    0.371    0.000    5.675    0.000 <ipython-input-132-93560edf79a4>:28(randint)
        1    0.008    0.008    6.187    6.187 <string>:1(<module>)
        1    0.000    0.000    6.187    6.187 {built-in method builtins.exec}
        1    0.000    0.000    0.000    0.000 {method 'disable' of '_lsprof.Profiler' objects}
```

```
In [188]: %prun [ CythonForthExample.randint() for _ in range(1000000) ]
```

```
         3000004 function calls in 2.831 seconds

   Ordered by: internal time

   ncalls  tottime  percall  cumtime  percall filename:lineno(function)
  1000000    1.425    0.000    1.425    0.000 {_cython_magic_55c22c8ddb4d3f9b7a2c470dde3236f5.
  1000000    0.757    0.000    2.182    0.000 <ipython-input-183-7bbfd5fb3182>:4(__next__)
        1    0.397    0.397    2.823    2.823 <string>:1(<listcomp>)
  1000000    0.244    0.000    2.426    0.000 <ipython-input-132-93560edf79a4>:28(randint)
        1    0.008    0.008    2.831    2.831 <string>:1(<module>)
        1    0.000    0.000    2.831    2.831 {built-in method builtins.exec}
        1    0.000    0.000    0.000    0.000 {method 'disable' of '_lsprof.Profiler' objects}
```

$\Longrightarrow$ the Cython version is twice as fast as the pure-Python version. We can still improve this, I am sure.

We can again check for the mean and the variance of the generated sequence. Mean should be $\frac{1}{2} = 0.5$ and variance should be $\frac{(b-a)^2}{12} = \frac{1}{12} = 0.08333\ldots$:

```
In [189]: shape = (400, 400)
          image = ForthExample.float_samples(shape)
          np.mean(image), np.var(image)
```

```
Out[189]: (0.49923488385706588, 0.083302180051650965)
```

This Python hand-written Mersenne twister is of course slower than the previous PRNG defined above (combined Multiple Recursive Generator, simple Multiple Recursive Generator, and the simple Linear Recurrent Generator):

```
In [190]: %timeit SecondExample.float_samples(shape)
          %timeit ThirdExample.float_samples(shape)
          %timeit MRG32k3a.float_samples(shape)
          %timeit ForthExample.float_samples(shape)
```

```
1 loop, best of 3: 224 ms per loop
1 loop, best of 3: 895 ms per loop
1 loop, best of 3: 1.28 s per loop
1 loop, best of 3: 1.19 s per loop
```

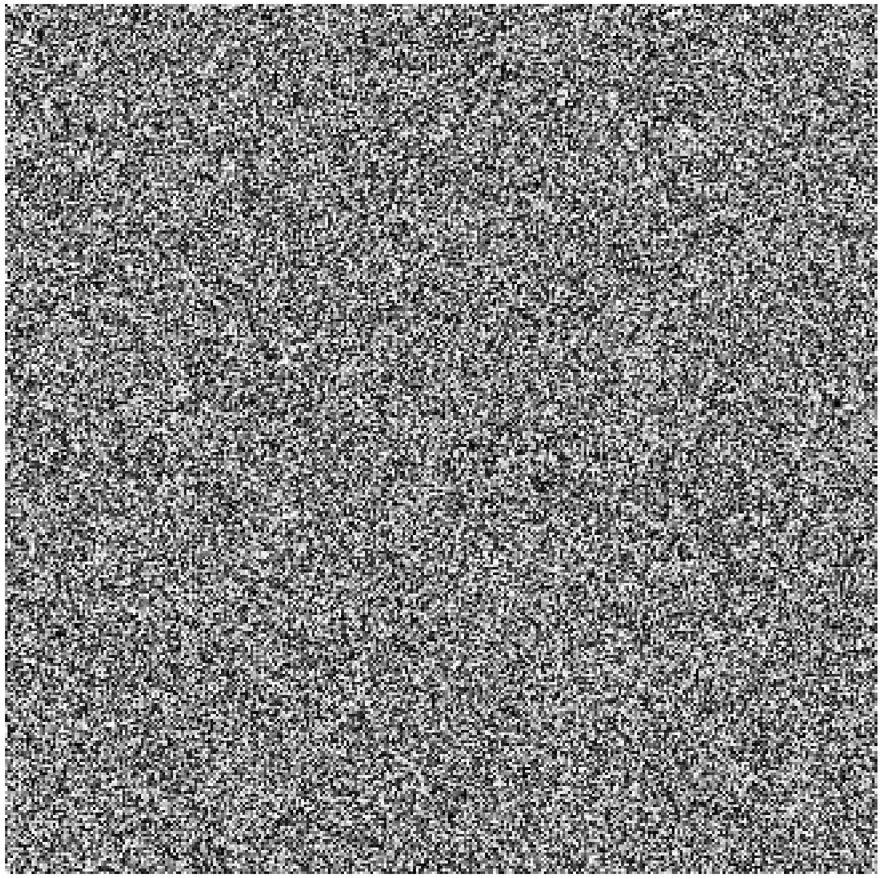
That's not too bad, for $400 \times 400 = 160000$ samples, but obviously it is incredibly slower than the optimized PRNG found in the `numpy.random` package.

```
In [191]: %timeit numpy.random.random_sample(shape)
```

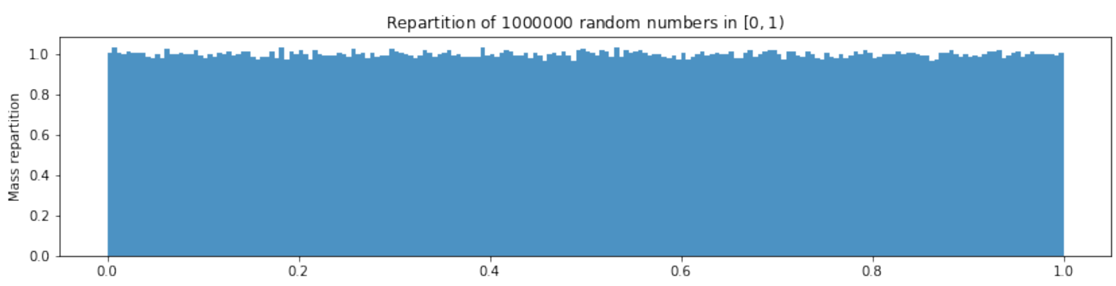
```
1000 loops, best of 3: 1.52 ms per loop
```

A good surprise is that this implementation Mersenne appears faster than the combined MRG of order $k = 3$ (i.e., `MRG32k3a`).

```
In [192]: showimage(image)
```



```
In [193]: plotHistogram(ForthExample, 1000000, 200)
```



2.8 Conclusion

Well, that's it, I just wanted to implement a few Pseudo-Random Number Generators, and compare them.

I should finish the job: - implement a test for "randomness", and check the various PRNG I implemented against it, - use these various `rand()` functions (uniform in $[0,1)$) to generate other distributions.

---

3 Generating samples from other distributions

So far, I implemented some PRNG, which essentially give a function `rand()` to produce float number uniformly sampled from $[0,1)$.

Let use it to generate samples from other distributions.

```
In [194]: def newrand():
              """Create a new random function rand()."""
              mersenne = MersenneTwister()
              rand = mersenne.rand
              return rand

          rand = newrand()
```

We will need an easy way to visualize the repartition of samples for the distributions defined below.

```
In [195]: def plotHistogramOfDistribution(distr, nb=10000, bins=200):
              numbers = [ distr() for _ in range(nb) ]
              plt.figure(figsize=(14, 3))
              plt.hist(numbers, bins=bins, normed=True, alpha=0.8)
              plt.xlabel("Random numbers from function %s" % distr.__name__)
              plt.ylabel("Mass repartition")
              plt.title("Repartition of ${}$ random numbers".format(nb))
              plt.show()
```

3.1 Bernoulli distribution

It is the simplest example, $X \in \{0,1\}$, $\mathbb{P}(X=0)=p$ and $\mathbb{P}(X=1)=1-p$ for some parameter $p \in [0,1]$.

```
In [196]: def bernoulli(p=0.5):
              """Get one random sample X ~ Bern(p)."""
              assert 0 <= p <= 1, "Error: the parameter p for a bernoulli distribution has to b
              return int(rand() < p)
```

In [197]:
```
print([ bernoulli(0.5) for _ in range(20) ])
print([ bernoulli(0.1) for _ in range(20) ])  # lots of 0
print([ bernoulli(0.9) for _ in range(20) ])  # lots of 1
```

```
[1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1]
[0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

We can quickly check that the frequency of 1 in a large sample of size n will converge to p as $n \to +\infty$:

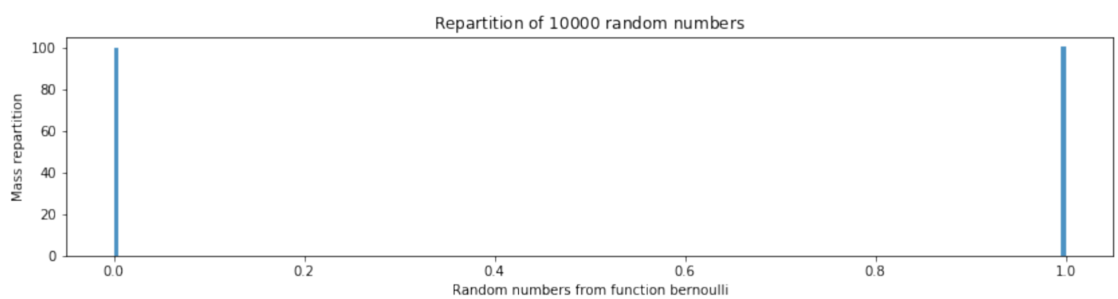
In [198]:
```
def delta_p_phat_bernoulli(p, nb=100000):
    samples = [ bernoulli(p) for _ in range(nb) ]
    return np.abs(np.mean(samples) - p)
```

In [199]:
```
print(delta_p_phat_bernoulli(0.5))
print(delta_p_phat_bernoulli(0.1))
print(delta_p_phat_bernoulli(0.9))
```

```
0.00089
0.00241
0.0017
```

That's less than 1% of absolute error, alright.

In [200]:
```
plotHistogramOfDistribution(bernoulli)
```

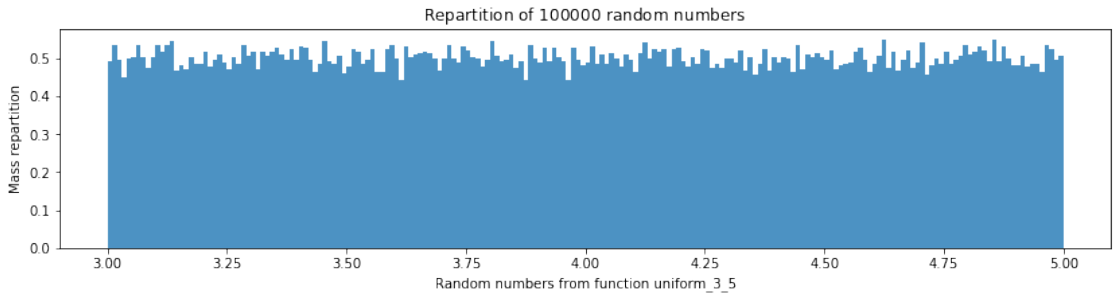


3.2 Uniform distribution on $[a, b)$, for floats and integers

This one is obvious too:

In [201]:
```
def uniform(a, b):
    """Uniform float number in [a, b)."""
    assert a < b, "Error: for uniform(a, b), a must be < b."
    return a + (b - a) * rand()
```

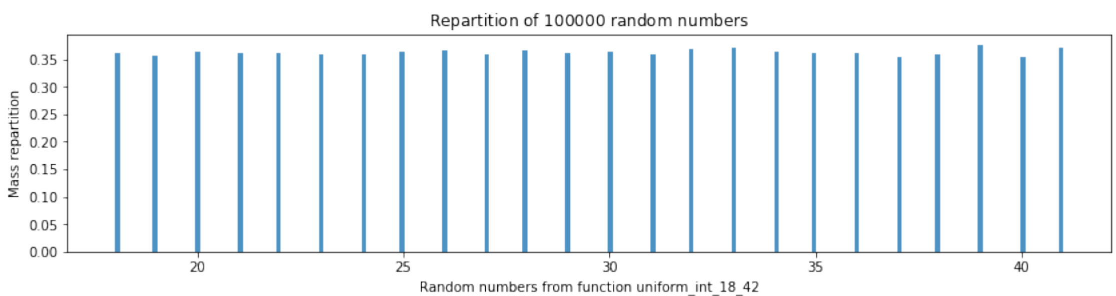
```
In [202]: def uniform_3_5():
              return uniform(3, 5)

          plotHistogramOfDistribution(uniform_3_5, 100000)
```



For integers, it is extremely similar:

```
In [203]: def randint(a, b):
              """Uniform float number in [a, b)."""
              assert a < b, "Error: for randint(a, b), a must be < b."
              assert isinstance(a, int), "Error: for randint(a, b), a must be an integer."
              assert isinstance(b, int), "Error: for randint(a, b), a must be an integer."
              return int(a + (b - a) * rand())
```

```
In [204]: def uniform_int_18_42():
              return randint(18, 42)

          plotHistogramOfDistribution(uniform_int_18_42, 100000)
```



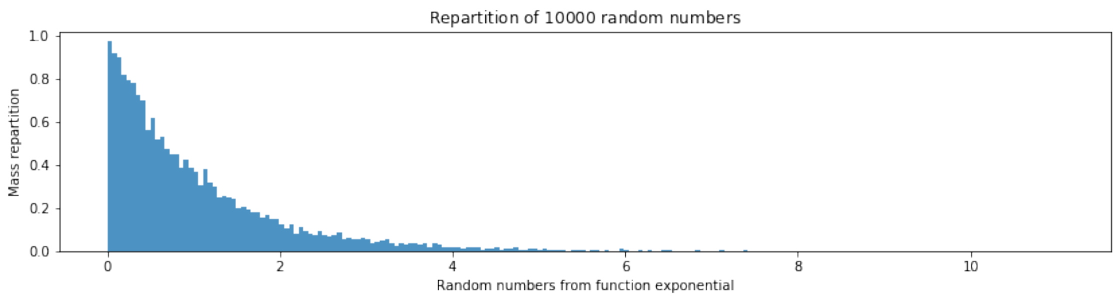
3.3 Exponential distribution

If $X \sim \mathrm{Exp}(\lambda)$, $F(x) = 1 - \mathrm{e}^{-\lambda x}$, and so $F^{-1}(u) = -\frac{1}{\lambda}\ln(1-u)$. The inversion method is easy to apply here:

```
In [205]: from math import log

          def exponential(lmbda=1):
              """Get one random sample X ~ Exp(lmbda)."""
              assert lmbda > 0, "Error: the parameter lmbda for exponential(lmbda) must be > 0
              u = rand()  # 1 - u ~ U([0, 1]), so u and 1 - u follow the same distribution
              return -(1.0 / lmbda) * log(u)
```

The resulting histogram has the shape we know as "exponential":

```
In [206]: plotHistogramOfDistribution(exponential)
```



We can compare its efficiency with `numpy.random.exponential()`, and of course it is slower.

```
In [207]: %timeit [ exponential(1.) for _ in range(10000) ]
          %timeit [ np.random.exponential(1.) for _ in range(10000) ]  # about 50 times slower

10 loops, best of 3: 86.3 ms per loop
100 loops, best of 3: 16.3 ms per loop
```

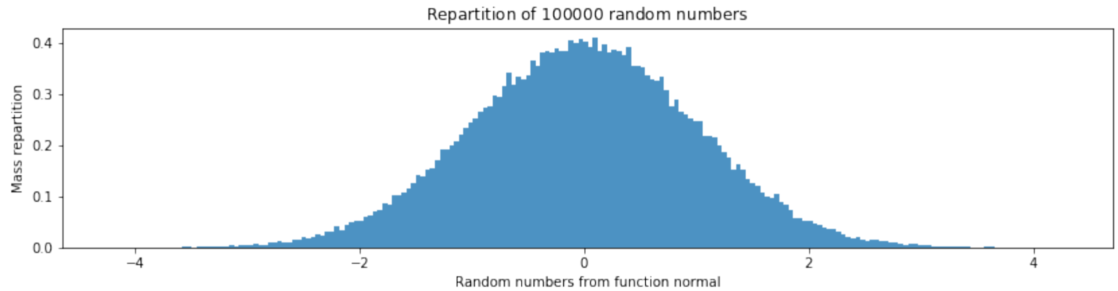
3.4 Gaussian distribution (normal)

By using the Box-Muller approach, if $U_1, U_2 \sim U(0,1)$ are independent, then setting $X = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $Y = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ leads to $X, Y \sim N(0,1)$.

Then $Z = \mu + \sigma * X$ will be distributed according to the Gaussian distribution of *mean* μ and *variance* $\sigma > 0$: $Z \sim N(\mu, \sigma)$.

```
In [208]: from math import sqrt, cos, pi

          def normal(mu=0, sigma=1):
              """Get one random sample X ~ N(mu, sigma)."""
              assert sigma > 0, "Error: the parameter sigma for normal(mu, sigma) must be > 0.
              u1, u2 = rand(), rand()
              x = sqrt(- 2 * log(u1)) * cos(2 * pi * u2)
              return mu + sigma * x
```

```
In [209]: plotHistogramOfDistribution(normal, 100000)
```



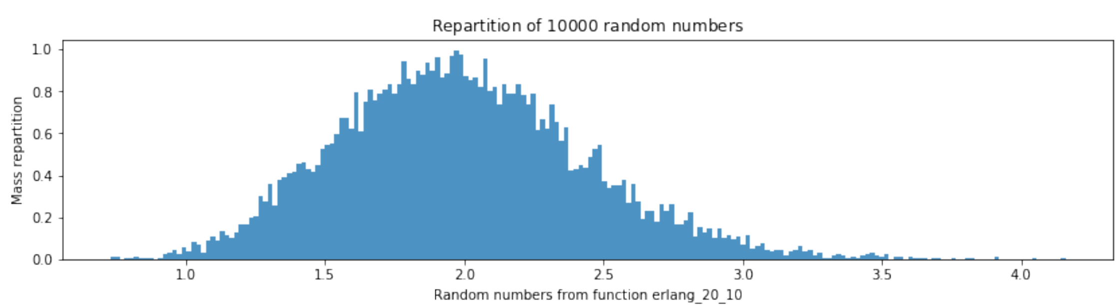
We can compare its efficiency with `numpy.random.normal()`, and of course it is slower.

```
In [210]: %timeit [ normal(0, 1) for _ in range(10000) ]
          %timeit np.random.normal(0, 1, 10000)  # 550 times quicker! oh boy!

1 loop, best of 3: 166 ms per loop
1000 loops, best of 3: 490 ţs per loop
```

3.5 Erlang distribution

If $X \sim \mathrm{Erl}(m, \lambda)$, then it simply is the sum of $m \in \mathbb{N}^*$ *iid* exponential random variables $Y_i \sim \mathrm{Exp}(\lambda)$.

```
In [211]: def erlang(m=1., lmbda=1.):
              """Get one random sample X ~ Erl(m, lmbda)."""
              assert m > 0, "Error: the parameter m for erlang(m, lmbda) must be > 0."
              assert lmbda > 0, "Error: the parameter lmbda for erlang(m, lmbda) must be > 0."
              return - 1. / lmbda * sum(log(rand()) for _ in range(int(m)) )

In [212]: def erlang_20_10():
              return erlang(20, 10)

          plotHistogramOfDistribution(erlang_20_10)
```

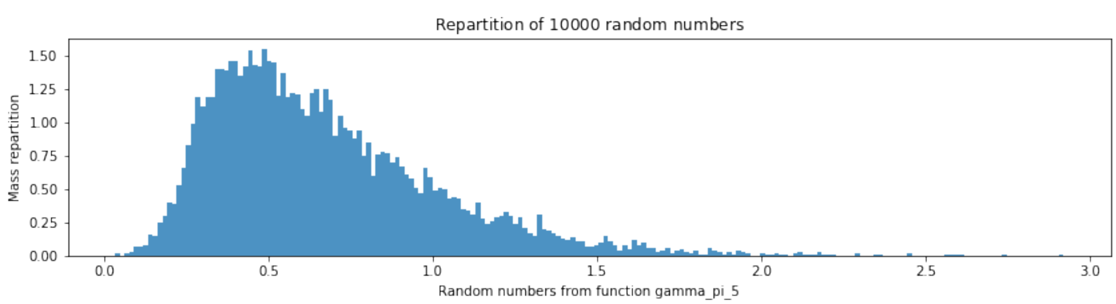


3.6 Gamma distribution

The algorithm is more complicated. The pdf of $X \sim \text{Gamma}(\alpha, \lambda)$ is $f(x) = x^{\alpha-1}\lambda^{\alpha}\mathrm{e}^{-\lambda x}/\Gamma(\alpha)$, for parameters $\alpha > 0, \lambda > 0$.

```
In [213]: def gamma(alpha=1., lmbda=1.):
              """Get one random sample X ~ Gamma(alpha, lmbda)."""
              assert alpha > 0, "Error: the parameter alpha for gamma(alpha, lmbda) must be > 0
              assert lmbda > 0, "Error: the parameter lmbda for gamma(alpha, lmbda) must be > 0
              if alpha <= 1:
                  x = gamma(alpha + 1., lmbda)
                  u = rand()
                  return x * (u ** (1. / alpha))
              else:
                  d = alpha - (1. / 3.)
                  oneByC = sqrt(9. * d)
                  c = 1. / oneByC
                  while True:
                      z = normal(0, 1)
                      if z > - oneByC:
                          v = (1. + c * z)**3
                          u = rand()
                          if log(u) < (.5 * (z**2)) + d*(v + log(v)):
                              break
                  return d * v / lmbda
```

```
In [214]: def gamma_pi_5():
              return gamma(pi, 5)

          plotHistogramOfDistribution(gamma_pi_5)
```



We can compare its efficiency with `numpy.random.gamma()`, and of course it is slower.

```
In [215]: %timeit [ gamma(pi, 5) for _ in range(10000) ]
          %timeit [ np.random.gamma(pi, 5) for _ in range(10000) ]  # 500 times quicker! oh boy
```

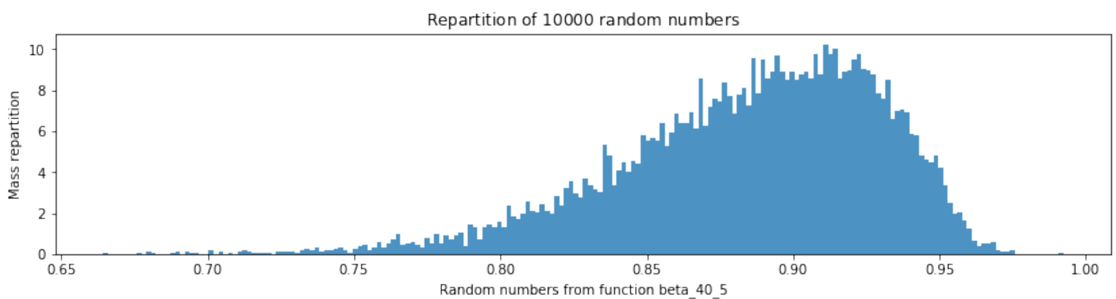
```
1 loop, best of 3: 285 ms per loop
10 loops, best of 3: 26.4 ms per loop
```

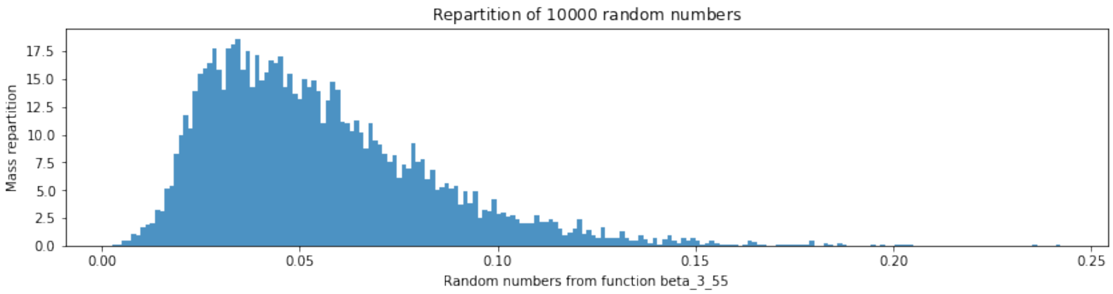
3.7 Beta distribution

By definition, a Beta distribution is straightforward to obtain as soon as we have a Gamma distribution: if $Y_1 \sim \mathrm{Gamma}(\alpha, 1)$ and $Y_2 \sim \mathrm{Gamma}(\beta, 1)$, then $X = \frac{Y_1}{Y_1+Y_2}$ follows $\mathrm{Beta}(\alpha, \beta)$.

```
In [216]: def beta(a=1., b=1.):
              """Get one random sample X ~ Beta(a, b)."""
              assert a > 0, "Error: the parameter a for beta(a, b) must be > 0."
              assert b > 0, "Error: the parameter b for beta(a, b) must be > 0."
              y1 = gamma(a, 1.)
              y2 = gamma(b, 1.)
              return y1 / float(y1 + y2)
```

```
In [217]: def beta_40_5():
              return beta(40, 5)

          plotHistogramOfDistribution(beta_40_5)

          def beta_3_55():
              return beta(3, 55)

          plotHistogramOfDistribution(beta_3_55)
```





We can compare its efficiency with `numpy.random.beta()`, and of course it is slower.

```
In [218]: %timeit [ beta(pi, 5*pi) for _ in range(1000) ]
          %timeit [ beta(5*pi, pi) for _ in range(1000) ]
          %timeit [ np.random.beta(pi, 5*pi) for _ in range(1000) ]  # 200 times quicker! oh bo
          %timeit [ np.random.beta(5*pi, pi) for _ in range(1000) ]  # 200 times quicker! oh bo
```

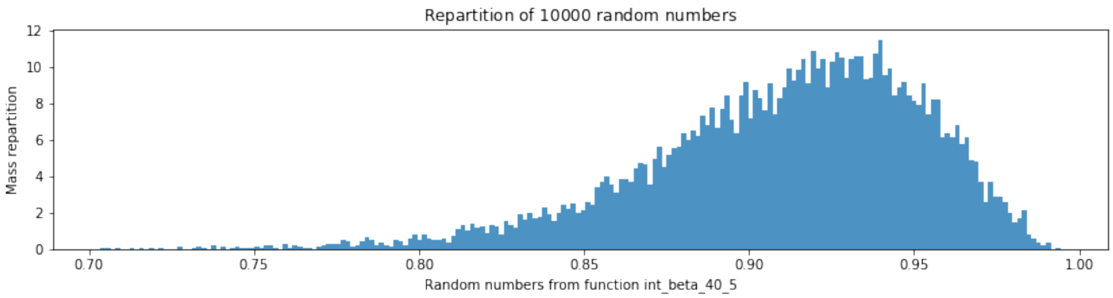
```
10 loops, best of 3: 68.2 ms per loop
10 loops, best of 3: 52.5 ms per loop
1000 loops, best of 3: 1.14 ms per loop
1000 loops, best of 3: 1.06 ms per loop
```

3.8 Integer Beta distribution

If $\alpha = m, \beta = n$ are integer, it is much simpler:

```
In [219]: def int_beta(m=1, n=1):
              """Get one random sample X ~ Beta(m, n) with integer parameters m, n."""
              assert m > 0, "Error: the parameter m for int_beta(m, n) must be > 0."
              assert n > 0, "Error: the parameter n for int_beta(m, n) must be > 0."
              us = [rand() for _ in range(m + n - 1)]
              return sorted(us)[m]  # inefficient to sort, but quick to write!
```

```
In [220]: def int_beta_40_5():
              return int_beta(40, 5)

          plotHistogramOfDistribution(int_beta_40_5)
```



We can again compare its efficiency with `numpy.random.beta()`, and of course it is slower, but this integer-specific implementation `int_beta()` is quicker than the generic `beta()` implementation.

```
In [221]: %timeit [ int_beta(40, 5) for _ in range(1000) ]
          %timeit [ int_beta(3, 55) for _ in range(1000) ]
          %timeit [ np.random.beta(40, 5) for _ in range(1000) ]  # 1500 times quicker! oh boy
          %timeit [ np.random.beta(3, 55) for _ in range(1000) ]  # 2000 times quicker! oh boy
```

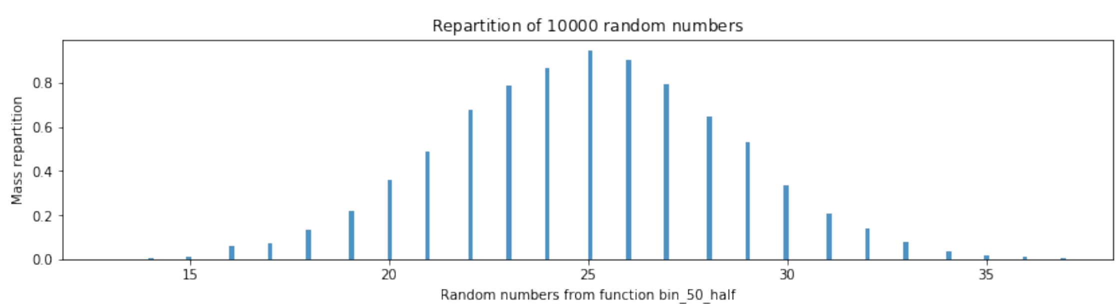
```
1 loop, best of 3: 340 ms per loop
1 loop, best of 3: 424 ms per loop
1000 loops, best of 3: 1.15 ms per loop
1000 loops, best of 3: 1.13 ms per loop
```

3.9 Binomial distribution

Very easy to obtain, by definition, from the sum of n Bernoulli distribution: if $Y_1, \dots, Y_n \sim \mathrm{Bern}(p)$, then $X = \sum_{i=1}^{n} Y_i \sim \mathrm{Bin}(n, p)$.

```
In [222]: def binomial(n=1, p=0.5):
              """Get one random sample X ~ Bin(n, p)."""
              assert 0 <= p <= 1, "Error: the parameter p for binomial(n, p) has to be in [0, 
              assert n > 0, "Error: the parameter n for binomial(n, p)  has to be in [0, 1]."
              return sum(bernoulli(p) for _ in range(n))
```

```
In [223]: def bin_50_half():
              return binomial(50, 0.5)

          plotHistogramOfDistribution(bin_50_half)
```



It is an integer distribution, meaning that $X \sim \mathrm{Bin}(n, p)$ always is $X \in \mathbb{N}$.

We can compare its efficiency with `numpy.random.binomial()`, and of course it is slower.

```
In [224]: %timeit [ binomial(10, 1. / pi) for _ in range(1000) ]
          %timeit [ np.random.binomial(10, 1. / pi) for _ in range(1000) ]  # 100 times quicker
```

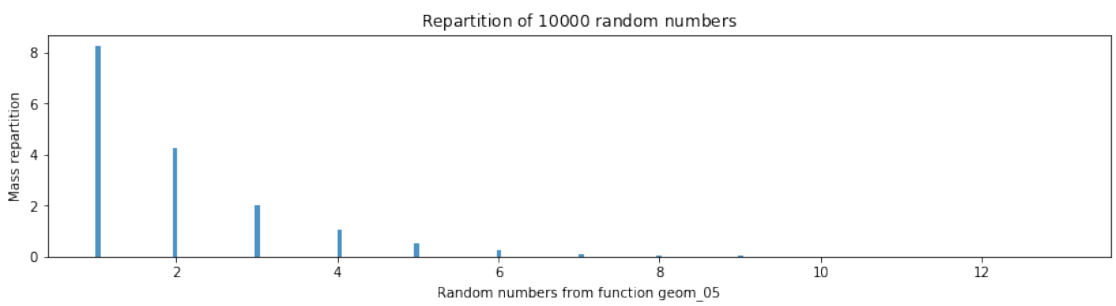
```
10 loops, best of 3: 99.5 ms per loop
1000 loops, best of 3: 1.86 ms per loop
```

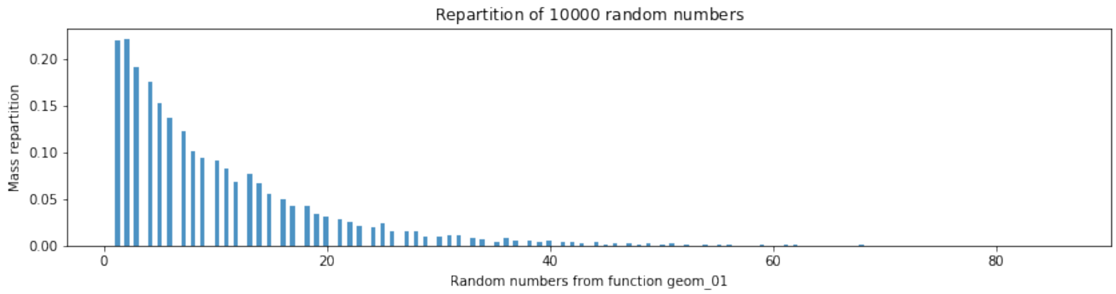
3.10 Geometric distribution

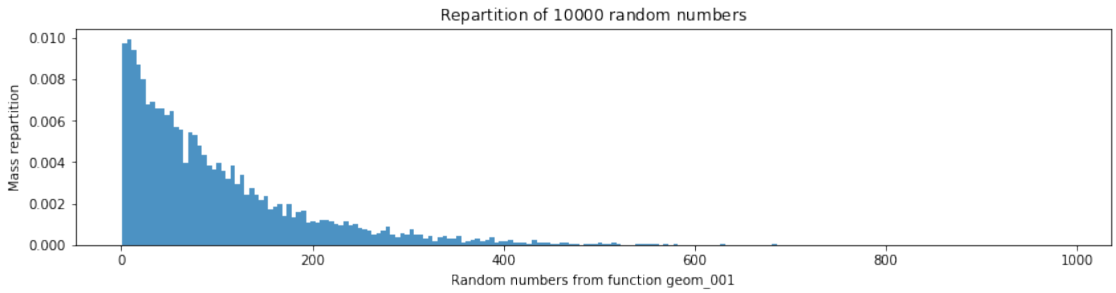
Again, it is very easy from the definition of a Geometric random variable.

```
In [225]: def geometric(p=0.5):
              """Get one random sample X ~ Geom(p)."""
              assert 0 <= p <= 1, "Error: the parameter p for binomial(n, p) has to be in [0, 1]."
              y = exponential(- log(1. - p))
              return 1 + int(y)
```

```
In [226]: def geom_05():
              return geometric(0.5)

          plotHistogramOfDistribution(geom_05)

          def geom_01():
              return geometric(0.1)

          plotHistogramOfDistribution(geom_01)

          def geom_001():
              return geometric(0.01)

          plotHistogramOfDistribution(geom_001)
```







We can compare its efficiency with `numpy.random.geometric()`, and of course it is slower.

```
In [227]: %timeit [ geometric(1. / pi) for _ in range(10000) ]
          %timeit [ np.random.geometric(1. / pi) for _ in range(10000) ]  # 50 times quicker, r
```

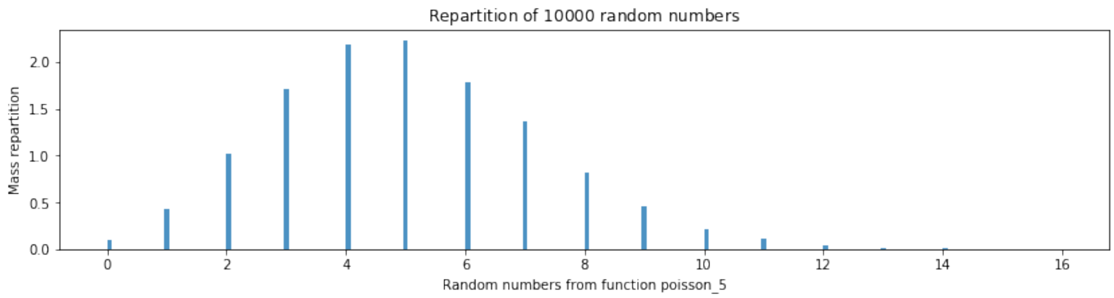
```
10 loops, best of 3: 87 ms per loop
100 loops, best of 3: 7.87 ms per loop
```

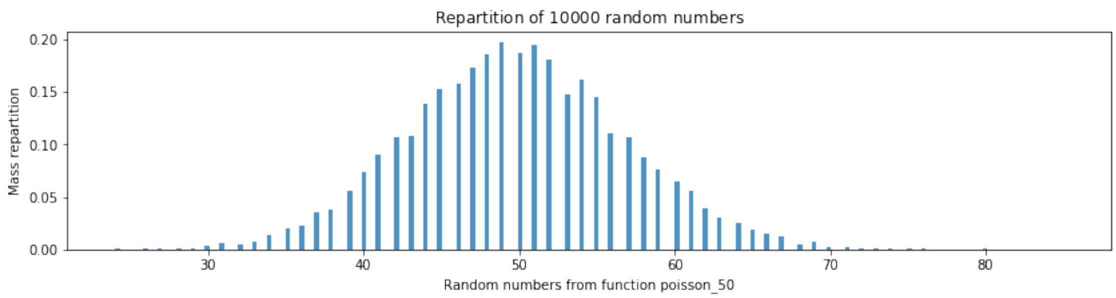
3.11 Poisson distribution

If $X \sim \mathrm{Pois}(\lambda)$, then its pdf is $f(n) = \frac{e^{-\lambda}\lambda^n}{n!}$. With the rejection method, and the close relationship between the Exponential and the Poisson distributions, it is not too hard to generate samples from a Poisson distribution if we know how to generate samples from a Exponential distribution.

```
In [228]: def poisson(lmbda=1.):
              """Get one random sample X ~ Poisson(lmbda)."""
              assert lmbda > 0, "Error: the parameter lmbda for poisson(lmbda) has to be > 0."
              n = 0
              a = 1
              while a >= exp(-lmbda):
                  u = rand()
                  a *= u
                  n += 1
              return n - 1
```

```
In [229]: def poisson_5():
              return poisson(5.)

          plotHistogramOfDistribution(poisson_5)

          def poisson_50():
              return poisson(50.)

          plotHistogramOfDistribution(poisson_50)
```





We can compare its efficiency with `numpy.random.poisson()`, and of course it is slower.

```
In [230]: %timeit [ poisson(12 * pi) for _ in range(1000) ]
          %timeit [ np.random.poisson(12 * pi) for _ in range(1000) ]  # 1000 times quicker! ol

1 loop, best of 3: 280 ms per loop
1000 loops, best of 3: 994 ţs per loop
```

3.12 Conclusion

Except the Gamma distribution, the algorithms presented above are easy to understand and to implement, and it was quick to obtain a dozen of the most common distributions, both continous (Exponential, Gaussian, Gamma, Beta) and discrete (Bernoulli, Binomial, Geometric, Poisson).

---

4 Generating vectors

Now that we have a nice Pseudo-Random Number Generator, using Mersenne twister, and that we have demonstrated how to use its `rand()` function to produce samples from the most common distributions, we can continue and explain how to produce vectors of samples.

For instance, one would need a `choice()` function to get a random sample from a list of n values, following any discrete distribution, or a `shuffle()` function to randomly shuffle a list.

4.1 Discrete distribution

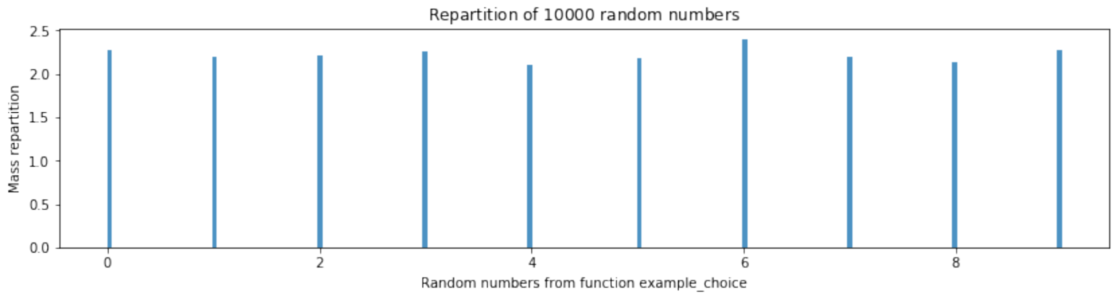
Let $p = [p_1, \dots, p_n] \in \mathbb{R}^n$ be a discrete distribution, meaning that $p_i > 0$ and $\sum_{i=1}^n p_i = 1$, then we can use the inverse-transform method to get a sample $i \in \{1, \dots, n\}$ with probability $\mathbb{P}(i = j) = p_j$.

```
In [231]: def discrete(p):
              """Return a random index i in [0..n-1] from the discrete distribution p = [p0,..
              n = len(p)
              assert n > 0, "Error: the distribution p for discrete(p) must not be empty!"
              assert all(0 <= pi <= 1 for pi in p), "Error: all coordinates of the distributio
              assert abs(sum(p) - 1) < 1e-9, "Error: the distribution p for discrete(p) does no
              u = rand()
              i = 0
              s = p[0]
              while i < n-1 and u > s:
                  i += 1
                  s += p[i]
              return i
```

Then it is easy to get *one* random sample from a list of values:

```
In [232]: def one_choice(values, p=None):
              """Get a random sample from the values, from the dsicrete distribution p = [p0,.
              if p is None:
                  return values[randint(0, len(values))]
              else:
                  return values[discrete(p)]
```

```
In [233]: def example_choice():
              return one_choice(range(10))

          plotHistogramOfDistribution(example_choice)
```



And it is also easy to generate many samples, with replacement.

```
In [234]: def choices_with_replacement(values, m=1, p=None):
              """Get m random sample from the values, with replacement, from the discrete dist
              if p is None:
                  return [ values[randint(0, len(values))] for _ in range(m) ]
              else:
                  return [ values[discrete(p)] for _ in range(m) ]
```

It is harder to handle the case without replacements. My approach is simple but slow: once a value is drawn, remove it from the input list, and update the discrete distribution accordingly. To be sure of not modifying the input list, I use `copy.copy()` to copy them.

```
In [235]: from copy import copy

          def choices_without_replacement(values, m=1, p=None):
              """Get m random sample from the values, without replacement, from the discrete d
              values = copy(values)
              if p is None:
                  samples = []
                  for _ in range(m):
                      i = randint(0, len(values))
                      samples.append(values[i])
                      del values[i]
              else:
                  p = copy(p)
                  samples = []
                  for _ in range(m):
                      i = discrete(p)
                      samples.append(values[i])
                      del values[i]
                      del p[i]
                      renormalize_cst = float(sum(p))
                      p = [ pi / renormalize_cst for pi in p ]
              return samples
```

```
In [236]: values = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
          p = [0.5, 0.1, 0.1, 0.1, 0.1, 0.02, 0.02, 0.02, 0.02, 0.02]
```

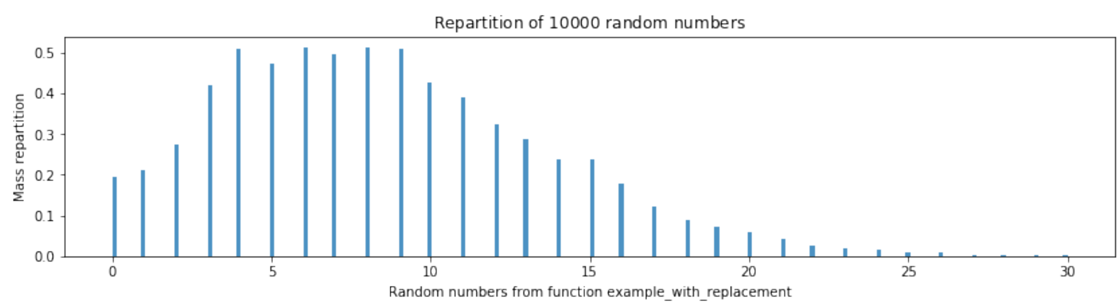
We can check that the input lists are not modified:

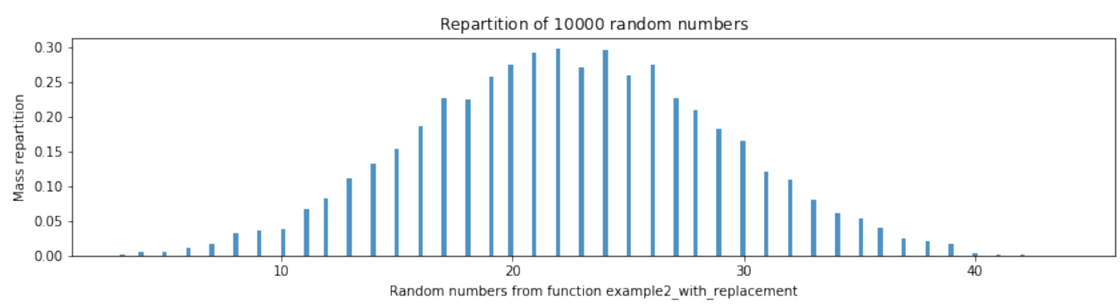
```
In [237]: print(choices_without_replacement(values, 5))
          print(choices_without_replacement(values, 5))
          print(choices_without_replacement(values, 5))

          print(choices_without_replacement(values, 5, p))
          print(choices_without_replacement(values, 5, p))
          print(choices_without_replacement(values, 5, p))
```

```
[7, 9, 1, 3, 2]
[0, 6, 5, 8, 1]
[1, 0, 9, 2, 6]
[0, 3, 4, 6, 1]
[0, 4, 3, 1, 2]
[0, 2, 1, 9, 3]
```

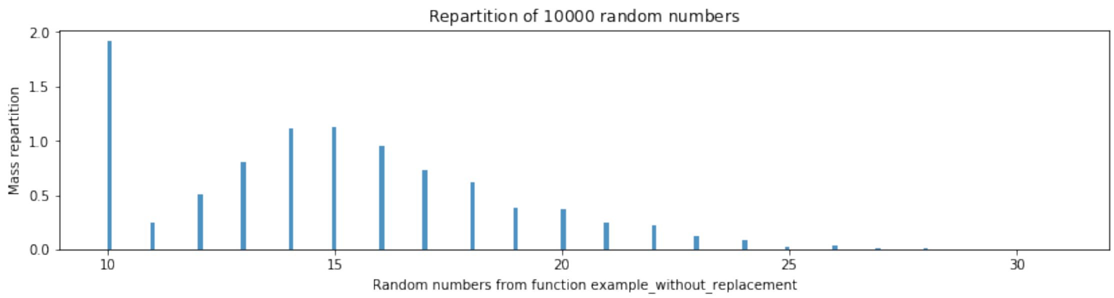
With an histogram, we can check that as a large weight is put on $0 =$ `values[0]`, the *sum* of $m = 5$ samples will be smaller if replacements are allowed (more chance to get twice a 0 value).

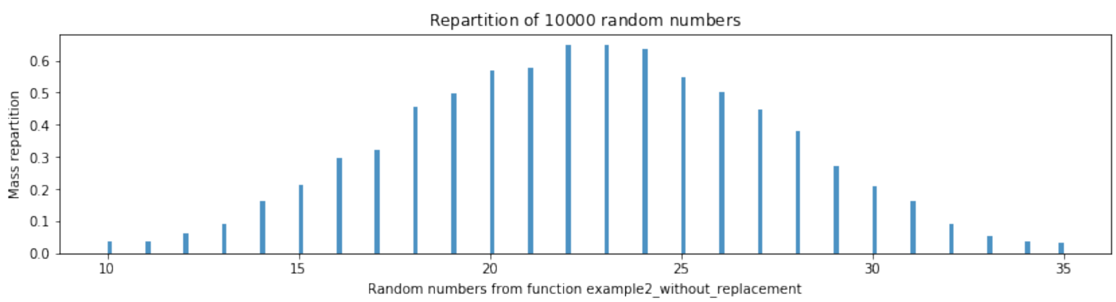
```
In [238]: def example_with_replacement():
              return np.sum(choices_with_replacement(values, 5, p))
          plotHistogramOfDistribution(example_with_replacement)

          def example2_with_replacement():
              return np.sum(choices_with_replacement(values, 5))
          plotHistogramOfDistribution(example2_with_replacement)
```





```
In [239]: def example_without_replacement():
              # this sum is at least >= 10 = 0 + 1 + 2 + 3 + 4 (5 smallest values)
              return np.sum(choices_without_replacement(values, 5, p))
          plotHistogramOfDistribution(example_without_replacement)

          def example2_without_replacement():
              # this sum is at least >= 10 = 0 + 1 + 2 + 3 + 4 (5 smallest values)
              return np.sum(choices_without_replacement(values, 5))
          plotHistogramOfDistribution(example2_without_replacement)
```





4.2 Generating a random vector uniformly on a n-dimensional ball

The acceptance-rejection method is easy to apply in this case. We use `uniform(-1, 1)` n times to get a random vector in $[0,1]^n$, and keep trying as long as it is not in the n-dim ball.

```
In [240]: def on_a_ball(n=1, R=1):
              """Generate a vector of dimension n, uniformly from the n-dim ball of radius R."
              rsquare = float('inf')
              Rsquare = R**2
```

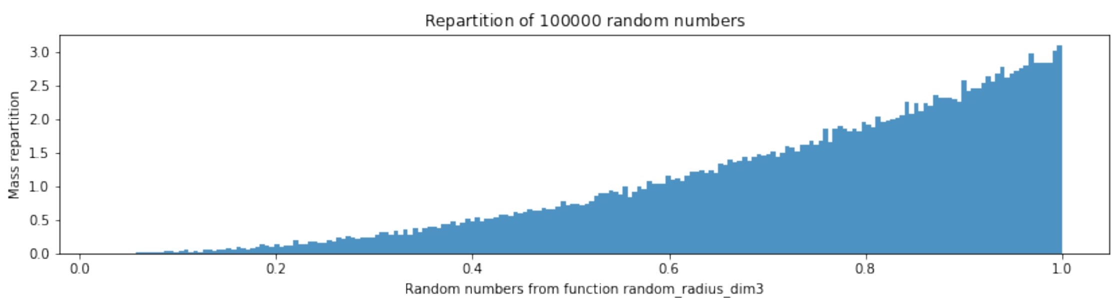
```
        while rsquare > Rsquare:
            values = [ uniform(-R, R) for _ in range(n) ]
            rsquare = sum(xi ** 2 for xi in values)
        return values
```

```
In [241]: print(on_a_ball(4, 1))
```

```
[0.047306195832788944, -0.77072893315926194, 0.37342280521988869, -0.42842319747433066]
```

The radius of such a vector can be plotted in a histogram.

```
In [242]: def random_radius_dim3():
              return sqrt(sum(xi**2 for xi in on_a_ball(3, 1)))

          plotHistogramOfDistribution(random_radius_dim3, 100000)
```

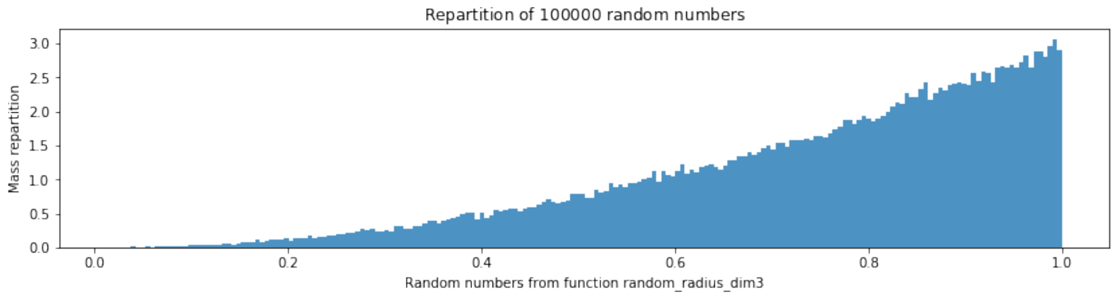


And similarly, if we normalize the values before returning them, to move them to the surface of the n-dimensional ball, then we get an easy way to sample a uniform *direction*:

```
In [243]: def on_a_sphere(n=1, R=1):
              """Generate a vector of dimension n, uniformly on the surface of the n-dim ball
              rsquare = float('inf')
              Rsquare = R**2
              while rsquare > Rsquare:
                  values = [ uniform(-1, 1) for _ in range(n) ]
                  rsquare = sum(xi ** 2 for xi in values)
              r = sqrt(rsquare)
              return [ xi / r for xi in values ]
```

All such samples have the same radius, but it can be interesting the see the smallest gap between two coordinates.

```
In [244]: def random_delta_dim3():
              return np.min(np.diff(sorted(on_a_sphere(3, 1))))

          plotHistogramOfDistribution(random_radius_dim3, 100000)
```



4.3 Generating a random permutation

The first approach is simple to write and understand, and it uses `choices_without_replacement([0..n-1], n)` with a uniform distribution p.

```
In [245]: def random_permutation(n=1):
              """Random permutation of [0..n-1], with the function choices_without_replacement
              return choices_without_replacement(list(range(n)), n)
```

```
In [246]: for _ in range(10):
              print(random_permutation(10))

[1, 4, 2, 8, 3, 7, 9, 6, 5, 0]
[9, 6, 0, 2, 1, 3, 8, 7, 5, 4]
[5, 1, 0, 6, 7, 9, 2, 3, 8, 4]
[1, 6, 4, 0, 5, 8, 2, 7, 3, 9]
[5, 4, 0, 2, 3, 6, 9, 8, 1, 7]
[0, 7, 6, 9, 1, 2, 3, 8, 5, 4]
[9, 7, 8, 4, 3, 0, 1, 2, 5, 6]
[4, 2, 0, 9, 1, 3, 7, 6, 5, 8]
[5, 9, 7, 8, 4, 2, 3, 1, 0, 6]
[0, 9, 7, 8, 2, 1, 5, 3, 4, 6]
```

It seems random enough!

To check this first implementation, we can implement the stupidest sorting algorithm, the "shuffle sort": shuffle the input list, as long as it is not correctly sorted.

```
In [247]: def is_sorted(values, debug=False):
              """Check if the values are sorted in increasing order, worst case is O(n)."""
              n = len(values)
              if n <= 1:
                  return True
              xn, xnext = values[0], values[1]
```

```
        for i in range(1, n + 1):
            if xn > xnext:
                if debug:
                    print("Values x[{}] = {} > x[{}+1] = {} are not in the good order!".
                return False
            if i >= n:
                return True
            xn, xnext = xnext, values[i]
        return True

    print(is_sorted([1, 2, 3, 4], debug=True))
    print(is_sorted([1, 2, 3, 4, 0], debug=True))
    print(is_sorted([1, 2, 5, 4], debug=True))
    print(is_sorted([1, 6, 3, 4], debug=True))
```

```
True
Values x[4] = 5 > x[5+1] = 0 are not in the good order!
False
Values x[5] = 4 > x[4+1] = 4 are not in the good order!
False
Values x[6] = 3 > x[3+1] = 3 are not in the good order!
False
```

We can easily apply a permutation, and return a shuffled version of a list of values.

```
In [248]: def apply_perm(values, perm):
              """Apply the permutation perm to the values."""
              return [values[pi] for pi in perm]

          def shuffled(values):
              """Return a random permutation of the values."""
              return apply_perm(values, random_permutation(len(values)))
```

Similarly, it is easy to shuffle in place a list of values.

```
In [249]: def shuffle(values):
              """Apply in place a random permutation of the values."""
              perm = random_permutation(len(values))
              v = copy(values)
              for (i, pi) in enumerate(perm):
                  values[i] = v[pi]
```

```
In [250]: def shuffle_sort(values):
              """Can you think of a more stupid sorting algorithm? or a shorter one?"""
              values = copy(values)
              while not is_sorted(values):
                  print(values)
                  shuffle(values)
              return values  # modified in place but also returned
```

In [251]: shuffle_sort([2, 1])

```
[2, 1]
```

Out[251]: [1, 2]

It is a **very** inefficient algorithm, but the fact that it works on small lists is enough to confirm that our algorithm to generate random permutations works fine.

In [252]:
```python
print(shuffle_sort(shuffled(list(range(3)))))
print(shuffle_sort(shuffled(list(range(4)))))
```

```
[0, 2, 1]
[0, 2, 1]
[2, 1, 0]
[1, 0, 2]
[0, 1, 2]
[1, 2, 0, 3]
[2, 3, 0, 1]
[0, 1, 2, 3]
```

We can think of another algorithm to generate a random permutation: - take n values $u_1, \ldots, u_n \sim U(0,1)$, - order them, - return the index of the sort.

In [253]:
```python
def random_permutation_2(n=1):
    """Random permutation of [0..n-1], by sorting n uniform values in [0,1]."""
    return list(np.argsort([rand() for _ in range(n)]))
```

In [254]:
```python
for _ in range(10):
    print(random_permutation_2(10))
```

```
[3, 8, 2, 1, 5, 6, 7, 0, 4, 9]
[7, 4, 1, 0, 5, 8, 2, 9, 3, 6]
[5, 7, 9, 6, 0, 2, 4, 1, 8, 3]
[3, 7, 1, 9, 6, 4, 5, 0, 2, 8]
[8, 7, 1, 5, 9, 4, 0, 3, 6, 2]
[2, 4, 3, 6, 8, 1, 0, 9, 5, 7]
[6, 5, 1, 8, 3, 4, 0, 2, 9, 7]
[3, 6, 2, 8, 0, 7, 4, 9, 5, 1]
[9, 4, 0, 1, 5, 6, 3, 8, 7, 2]
[0, 1, 6, 7, 9, 2, 8, 4, 5, 3]
```

It seems random enough too!
Let compare which of the two algorithms is the fastest:

In [255]:
```python
%timeit random_permutation(100)
%timeit random_permutation_2(100)
```

```
1000 loops, best of 3: 888 ţs per loop
1000 loops, best of 3: 754 ţs per loop
```

```
In [256]: %timeit random_permutation(10000)
          %timeit random_permutation_2(10000)
```

```
10 loops, best of 3: 94.5 ms per loop
10 loops, best of 3: 83.5 ms per loop
```

It seems that the first algorithm is slower, but this comes from the naively-written `choice_without_replacement()`, in fact we can implement it more efficiently.

```
In [257]: def random_permutation_3(n=1):
              """Random permutation of [0..n-1], with a smart implementation of choices_withou
              p = list(range(n))
              values = []
              for i in range(n):
                  j = randint(0, n - i)
                  values.append(p[j])
                  p[i], p[j] = p[j], p[i]
              return values
```

```
In [258]: for _ in range(10):
              print(random_permutation_3(10))
```

```
[0, 8, 0, 5, 3, 3, 2, 6, 7, 1]
[3, 7, 6, 4, 4, 3, 6, 2, 7, 5]
[5, 4, 3, 1, 4, 2, 5, 0, 6, 8]
[0, 1, 6, 4, 6, 3, 5, 0, 1, 7]
[1, 6, 3, 4, 3, 4, 5, 2, 6, 1]
[7, 5, 3, 6, 6, 7, 1, 2, 5, 0]
[3, 2, 7, 6, 7, 2, 5, 4, 3, 8]
[2, 0, 2, 0, 4, 3, 5, 6, 7, 1]
[6, 0, 2, 4, 2, 3, 4, 0, 7, 6]
[1, 1, 1, 1, 5, 1, 3, 0, 2, 7]
```

```
In [259]: %timeit random_permutation(1000)
          %timeit random_permutation_2(1000)
          %timeit random_permutation_3(1000)
          %timeit numpy.random.permutation(1000)

          %timeit random_permutation(10000)
          %timeit random_permutation_2(10000)
          %timeit random_permutation_3(10000)
          %timeit numpy.random.permutation(10000)  # About 1000 times slower! Oh boy!!
```

```
100 loops, best of 3: 9.28 ms per loop
100 loops, best of 3: 7.43 ms per loop
100 loops, best of 3: 9.32 ms per loop
The slowest run took 4.46 times longer than the fastest. This could mean that an intermediate r
10000 loops, best of 3: 23.3 ţs per loop
10 loops, best of 3: 95.7 ms per loop
10 loops, best of 3: 72.1 ms per loop
10 loops, best of 3: 87 ms per loop
10000 loops, best of 3: 173 ţs per loop
```

Hoho, not so sure on small lists... But for larger values of n, the second implementation of the first algorithm wins:

```
In [260]: %timeit random_permutation(100000)
          %timeit random_permutation_2(100000)
          %timeit random_permutation_3(100000)
          %timeit numpy.random.permutation(100000)  # About 1000 times slower! Oh boy!!
```

```
1 loop, best of 3: 1.62 s per loop
1 loop, best of 3: 749 ms per loop
1 loop, best of 3: 983 ms per loop
100 loops, best of 3: 2.03 ms per loop
```

And the second algorithm wins, as it uses the optimized `numpy.argsort()` function as its core operator.

Conclusion

This last part presented how to generate from any discrete distribution, and then two algorithms to generate a random permutation, uniformly sampled from Σ_n (set of $n!$ permutations of $\{0, \ldots, n-1\}$).

That's it for today, folks!